

RATIONALIZABLE SCREENING AND DISCLOSURE UNDER UNAWARENESS*

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Abstract

This paper analyzes a principal-agent procurement problem in which the principal is unaware of events affecting the agent’s marginal costs. Since she does not conceive of all relevant events, her planned menu of contracts may be suboptimal. Communication arises naturally as some types of the agent may have an incentive to make the principal aware of some of those events before a contract menu is offered. The menu must not only reflect the principal’s change in awareness, however: She also learns about the agent’s types, as not all of them may have incentives to raise her awareness. We capture this reasoning through an extensive-form version of cautious rationalizability with beliefs on marginal cost types restricted to logconcavity and “reverse” Bayesianism (Karni and Vierø, 2013). We show that if initially the principal is only unaware of some low marginal cost types, she is not made aware of all types and there is *bunching at the top*. If the principal is only unaware of some high marginal cost types, then she becomes aware of all types. Thus, the principal is happily made aware of inefficiencies but kept tacitly in the dark about some efficiencies. In any case, the principal offers an optimal menu of contracts for all types of which she is or has become aware.

Keywords: Hidden information; screening; contract design; principal-agent model; procurement; games with unawareness; extensive-form rationalizability; Δ -rationalizability; iterated admissibility; common belief in rationality.

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1 Introduction

In many real-life contracting situations, the agent hired by a principal is much more experienced in the task than the principal. This is often the reason why an agent is hired in the first place. Examples are an economics professor that hires a contractor to remodel a house, or a firm hiring an investment bank to prepare for stock offering. Thus, it is conceivable that the agent is aware of relevant events that the principal has not thought about at all at the time of writing the contract. In other situations, contracting is complex and involves trade secrets or research & development, making it natural to assume that the principal may not have conceived all relevant events when offering contracts. For instance, when procuring novel, highly specialized assets such as security or weapon systems, government agencies may not be aware of all technological details, let alone the contractors' cost structure. As a result of the principal's limited awareness, the contractors may find the offers to be unappealing or overly generous. Similarly, a monopoly regulator may not be privy to certain details of the monopolist's technology, rendering the regulation standards inadequate.

The standard approach to contract design under hidden information (Mirrlees, 1971; Mussa and Rosen, 1978; Baron and Myerson, 1982; Maskin and Riley, 1984) poses that, while the principal does not observe the agent's actual type, there is implicitly common awareness of all possible types and of their distribution among the principal and the agent. Consequently, the principal offers a menu of contracts that maximizes her expected utility subject to self-selection constraints. Typically, the menu is designed precisely in such a way that the agent accepts the offer and picks the contract within the menu that is meant for his actual type. The situations described above, although both common and relevant, are thus excluded from the realm of standard contract theory.

This paper analyzes a procurement problem where the principal is unaware of some of the agent's marginal cost types, in the context of extensive-form games with unawareness (Heifetz, Meier, and Schipper, 2013; see Halpern and Rêgo, 2014, and Feinberg, 2020, for alternatives). We posit that the agent has one of various finite numbers of marginal cost types. While screening with a finite number of types has been studied in the literature (Spence, 1980; Cooper, 1984; Maskin and Riley, 1984; Matthews and Moore, 1987), we allow simultaneously for different *sets* of finite types to accommodate the principal's unawareness. Before the actual screening problem, the agent may choose to change the principal's awareness. Indeed, communication arises naturally in the problems we study, as some agent types may have incentives to raise the principal's awareness of marginal cost types. This does not only change the principal's awareness, however: She also needs to reason about which types of the agent of which she is now aware have incentives to raise her awareness in the first place. We capture this reasoning through a version of extensive-form rationalizability that captures a notion of caution and further restrictions.

In order to link the principal's beliefs over marginal cost types across different awareness levels, we impose "reverse" Bayesian updating, axiomatized for single-person decision making by Karni and Vierø (2013) and Dominiak and Tserenjigmid (2018). In our context, reverse Bayesianism means that the relative likelihood of any marginal cost types θ and θ' that the principal conceived initially remains the same upon becoming aware of an additional type θ'' , as long as θ and θ' are not ruled out. To the best of our knowledge, this property has not been used in prior work on games with unawareness.

We further restrict beliefs over agent types to be logconcave, which implies that hazard rates are nondecreasing—a standard assumption in the screening literature that reduces the number of binding incentive compatibility constraints (see Mussa and Rosen, 1978; Baron and Myerson, 1982; Maskin and Riley, 1984; Matthews and Moore, 1986). In our context, logconcavity represents the common belief that the principal’s marginal beliefs over marginal cost types are unimodal in their support. Although various restrictions on first-order beliefs have been previously considered in different versions of rationalizability (Battigalli and Siniscalchi, 2003; Battigalli, 2006; Battigalli and Friedenberg, 2012; Battigalli and Prestipino, 2013), we are not aware of this particular restriction being previously explored in rationalizability.

We also rely on a tie-breaking assumption that is incorporated into first-order beliefs of rationalizability. Altogether, our solution concept has features of extensive-form rationalizability (Pearce, 1984; Battigalli, 1997; Heifetz, Meier, and Schipper, 2013); it captures a notion of caution (Heifetz, Meier, and Schipper, 2020) similar to iterated admissibility (Brandenburger, Friedenberg, and Keisler, 2008); and finally, it incorporates additional restrictions akin to Δ -rationalizability, extensive-form best response sets, or self-admissible sets (see Battigalli and Siniscalchi, 2003; Battigalli and Friedenberg, 2012; Battigalli and Prestipino, 2013; Brandenburger, Friedenberg, and Keisler, 2008; Brandenburger and Friedenberg, 2010). We are not aware that the screening problem has been solved previously with a rationalizability notion.

We show that if the principal is unaware only of events that increase marginal costs, then all types of the agent have an incentive to fully raise her awareness. Consequently, the principal offers a menu of contracts that maximizes her expected payoff with respect to a full support belief over agent types subject to incentive compatibility and participation constraints. If the principal is unaware only of events that decrease marginal costs, however, she is not made fully aware. None of the types of agent of which she is initially aware have incentives to raise her awareness. If a type of the agent of which she is unaware prefers to raise her awareness, he will not do so to a level that includes his own type. Consequently, there is *bunching of unaware types at the top*. In any case, the principal offers an optimal menu of contracts to all types of which she is or has become aware.

Although games with unawareness are relatively new, this is not the first paper to study contracting under unawareness. von Thadden and Zhao (2012) study a principal-agent moral hazard problem in which the principal is aware of actions of which the agent is unaware. When contemplating whether or not to make the agent aware, the principal faces a trade-off between getting a better action and saving on information rents due to additional incentive compatibility constraints. Auster (2013) studies a principal-agent moral hazard problem in which the principal is aware of contingencies of which the agent is unaware but whose realization is informative about the agent’s actions. In the optimal contract, the principal faces a trade-off between exploiting the agent’s unawareness and using said contingencies in order to provide incentives. Filiz-Ozbay (2012) studies a risk neutral insurer who is aware of some contingencies that the insuree is unaware. The insurer has an incentive to mention only some contingencies in a contract while be silent on others.

In all of the papers above, the principal has a higher awareness level than the agent, in contrast to our paper. Auster and Pavoni (2019) and Lei and Zhao (2019) feature an agent with higher awareness level than the principal but in the context of optimal delegation. In Auster and Pavoni (2019), the agent is aware of both the set of his actions and their performance, and only reveals extreme actions. In Lei and Zhao (2019), the agent is aware of some contingencies that

the principal is not, and only partially reveals such contingencies. Principals who are unaware of more contingencies delegate a large set of projects. Finally, Ma and Schipper (2012) show that the arguments for welfare irrelevance of indescribable contingencies by Maskin and Tirole (1999) extend to persistent asymmetric information settings but not to asymmetric unawareness in a buyer-seller model. Grant, Kline, and Quiggin (2012) discuss disagreements arising from asymmetric awareness in contracting.

Since communication arises naturally in our principal-agent problem as some types of the agent may have an incentive to raise the principal's awareness of the existence of some marginal cost types, our paper is related to disclosure in games with unawareness. Heifetz, Meier, and Schipper (2020) show that the unraveling argument breaks down in disclosure games à la Milgrom and Roberts (1986) when receivers may be unaware of some signalling dimension. In such a case, the receiver is unable to infer anything about the sender's type from the absence of a signal. This has been experimentally tested in Li and Schipper (2018). Schipper and Woo (2019) apply this insight to electoral campaigning that allows them to discuss microtargeting of voters and negative campaigning. Carvajal, Rostek, Schipper, and Sublet (2020) study the effect of disclosure of awareness before IPOs and show that it has the opposite effect to the disclosure of information.

The closest work to ours on adverse selection without unawareness are Pram (2020) and Ali, Lewis, and Vasserman (2019). Both papers consider a screening problem in which the agent can disclose verifiable evidence about his type before the principal commits to a mechanism. Pram (2020) characterizes environments in which verifiable evidence is welfare improving, namely whenever in the mechanism without disclosure some types would be excluded. Ali, Lewis, and Vasserman (2019) also find that the agent can benefit from prior disclosure of evidence but it depends on that partial disclosure being feasible. Our setting can be also viewed as one with disclosure by the agent prior to the principal committing to the mechanism. However, instead of disclosure of information, we consider disclosure of awareness. In some sense, disclosure of awareness is the opposite of disclosure of information, as awareness of more types is akin to increasing uncertainty about the possible types that the principal faces.

Somewhat related is Sher and Vohra (2015), who also consider disclosure in a price discrimination problem of a monopolist. Yet, their monopolist commits to a mechanism with evidence-contingent prices; i.e., disclosure occurs after commitment to the mechanism. Such a timing is much less compelling in the face of unawareness, as the principal is unaware of potential evidence that could be disclosed. Hidir and Vellodi (2019) study buyer's cheap-talk prior trade and buyer optimal market segmentation consistent with their information revelation.

As we use a rationalizability notion with belief restrictions as solution concept, our work is also related to the mechanism design literature on implementation in rationalizable strategies. The closest work to ours here is Ollár and Penta (2017), who study full implementation in rationalizable strategies with belief restrictions.

The paper is organized as follows: The next section outlines the model and our solution concept. In Section 3, we study the case in which the principal is initially unaware of lower marginal cost types only, both in a setting with three types and in a setting with an arbitrary number of finite types. Section 4 follows up with an analogous analysis for the case in which the principal is initially unaware of higher marginal cost types only. We conclude with a discussion in Section 5. The proofs of the main results and results that we require throughout the analysis are relegated to appendices.

2 Model

2.1 Screening Game with Unawareness

Consider a principal (P , “she”) who wants to procure $q \geq 0$ units of output from an agent (A , “he”). The principal’s utility of output is given by $v(q)$, with $v'(q) > 0$, $v''(q) < 0$ and $v(0) = 0$; we also impose the Inada condition $\lim_{q \rightarrow 0} v'(q) = +\infty$. The net utility or payoff for the principal from procuring q units from the agent in exchange for payment $t \geq 0$ is $u_P(q, t) = v(q) - t$. We assume that contracts are bounded: $q, t \leq b$ for some bound $b > 0$.

The agent’s marginal cost of production depends on the initial move of nature. Let $\bar{\Theta}$ be the nonempty finite set of all initial moves of nature; for simplicity, let $\bar{\Theta} = \{1, \dots, n\}$ for some natural number $n > 1$. To represent asymmetric unawareness of marginal costs, we consider subsets $\Theta \subseteq \bar{\Theta}$ that are “intervals”: $\theta \in \bar{\Theta}$ with $\min \Theta \leq \theta \leq \max \Theta$ implies $\theta \in \Theta$. Typically, we assume that the agent is aware of all marginal cost types in $\Theta_A = \bar{\Theta}$, no matter what is his type,¹ while the principal is initially unaware of some types: $\Theta_P \subsetneq \bar{\Theta}$.

The agent’s marginal cost of production is a type-dependent injective function $c : \bar{\Theta} \rightarrow \mathbb{R}_+$ to be specified later depending on the particular models. The profit of an agent of type θ from selling q units to the principal and collecting payment t is $u_A(q, t, \theta) = t - c(\theta)q$.

The game proceeds as follows: Nature draws the marginal cost type of the agent. The agent is privately informed about his type. He can then decide whether or not disclose the existence of some subset of marginal cost types to the principal. The principal’s initial awareness of marginal cost types, Θ_P , and the message by the agent determines her interim awareness and subsequent choice of contract menus. She cannot reason about types she does not conceive interim. To model this, we consider poorer descriptions of the game trees in which nature draws initially only among types of which the principal is aware. There is a forest of game trees, one tree for each possible space of draws of nature she could become aware after disclosure by the agent; see Figure 1 for a version with three trees. These trees are ordered by the richness of their set of moves of nature. Depending on disclosure, the principal lives in one of those trees as indicated by the information sets. For instance, if the agent remains silent, then her awareness is given by the lowest tree.

Figure 1 also illustrates two non-standard features of extensive-form games with unawareness. First, instead of one tree, there is a forest of trees representing possible views of moves by all players, including nature. Second, a player’s information set at a history may consist of histories in a lower tree, signifying the fact that players may be unaware of some moves. See Heifetz, Meier, and Schipper (2013) for details on extensive-form games with unawareness.

After disclosure, the principal offers a menu of contracts to the agent. He picks one of the contracts or the outside option, whose payoff is normalized to zero, and the game ends.²

¹Letting the agent’s awareness depend on his type as well would make it easier for the principal to infer the agent’s type from becoming aware. We do not allow this here as our goal is to focus on the effect of disclosing awareness only. See Pram (2020) and Ali, Lewis, and Vasserman (2019) for a screening problem with prior disclosure of information only.

²We could allow for the principal offering first an initial menu of contracts after which she might be made aware of additional marginal cost types of the agent and subsequently offer a possibly revised menu of contracts. This would not change our results in any essential way, so we opt for the simpler description of the game.

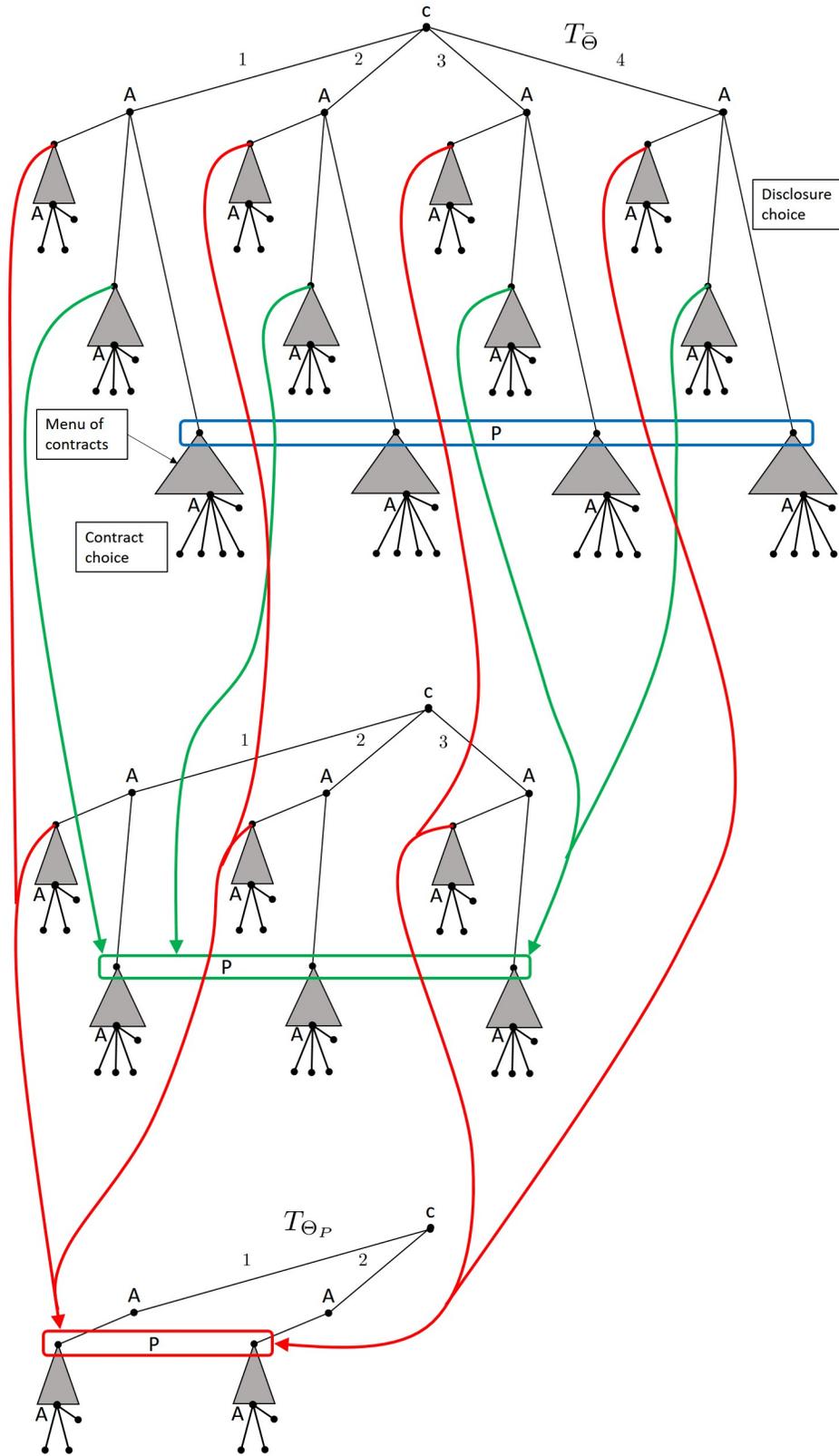


Figure 1: Game Form

2.2 Solution Concept

In order to define our solution, we define first (pure) strategies and belief systems. Formally, let H_i be player i 's information sets, for $i = A, P$, across all trees. A *strategy* of player i assigns to each of their information sets an action available at that information set, and is denoted by $(s_i(h_i))_{h_i \in H_i}$. Let S_i be the set of strategies of player i and S_{-i} represent the set of strategies of player i 's opponent. Note that the agent has perfect information everywhere; his information sets are singleton. The principal has exactly one information set in each tree, and her strategies assign a menu of contracts to each of her information sets.

A strategy s_i reaches information set h_i if there exists an opponent's strategy s_{-i} and move of nature such that the path induced by (s_i, s_{-i}) and the move of nature leads to the information set h_i . Denote by T_{h_i} the tree in which information set h_i is located. Similarly, we denote by Θ_{h_i} the space of moves of nature in the tree in which information set h_i is located. Finally, let s^T be the T -partial strategy restricted to information sets in tree T and any tree poorer than T , with S^T denoting the set of T -partial strategies.

Recall that the agent has perfect information about the move of nature. Thus, he forms beliefs only about the principal's strategy. For any finite set K , let $\Delta(K)$ denote the set of probability distributions on K . A *belief system* of the agent is given by:

$$\beta_A = (\beta_A(h_A))_{h_A \in H_A} \in \prod_{h_A \in H_A} \Delta(S_P^{T_{h_A}}),$$

which is a profile of beliefs—a belief $\beta_A(h_A) \in \Delta(S_P^{T_{h_A}})$ for each information set $h_A \in H_A$ about the principal's strategies in the T_{h_A} -partial game—with the following properties:

- (i) *Non-delusion*: $\beta_A(h_A)$ reaches h_A ; i.e., $\beta_A(h_A)$ assigns probability 1 to the set of strategy profiles of the principal that reach h_A .
- (ii) *Bayesianism*: If h_A precedes h'_A , then $\beta_A(h'_A)$ is the conditional belief derived from $\beta_A(h_A)$ whenever possible.

The principal forms beliefs both over strategies of the agent and moves of nature. Her *belief system* is given by a profile of beliefs:

$$\beta_P = (\beta_P(h_P))_{h_P \in H_P} \in \prod_{h_P \in H_P} \Delta(S_A^{T_{h_P}} \times \Theta_{h_P})$$

which has the following properties:

- (i) *Non-delusion*: $\beta_P(h_P)$ reaches h_P ; i.e., $\beta_P(h_P)$ assigns probability 1 to the set of strategy profiles of the agent and moves of nature that reach h_P .
- (ii) *Logconcavity*: Denote by $p_{\Theta_{h_P}} := \text{marg}_{\Theta_{h_P}} \beta_P(h_P)$ the marginal probability on moves of nature in Θ_{h_P} , and by $\text{supp } p_{\Theta_{h_P}} = \{\theta : p_{\Theta_{h_P}}(\theta) > 0\}$, the support of $p_{\Theta_{h_P}}$. Further, let $\kappa^{(i)}$ denote the order statistics of marginal costs of types in $\text{supp } p_{\Theta_{h_P}}$, and let $p_{\Theta_{h_P}}^i =$

$p_{\Theta_{h_P}}(\theta)$ if $\kappa^{(i)} = c(\theta)$. (Recall that the marginal cost function c is injective.) We require that $p_{\Theta_{h_P}}$ is logconcave; i.e., for all $i = 2, \dots, |\text{supp } p_{\Theta_{h_P}}| - 1$,

$$p_{\Theta_{h_P}}^i p_{\Theta_{h_P}}^{i+1} \geq p_{\Theta_{h_P}}^{i+1} p_{\Theta_{h_P}}^{i-1}.$$

- (iii) “Reverse” Bayesianism: Let $\Theta_{h_P} \subseteq \Theta_{h'_P}$. If β_P is such that the principal at h'_P cannot rule out any type in Θ_{h_P} from the agent disclosing $\Theta_{h'_P}$, then the marginals of the principal’s belief systems $p_{\Theta_{h_P}}$ and $p_{\Theta_{h'_P}}$ satisfy “reverse” Bayesianism: For all $\theta, \theta' \in \Theta_{h_P}$ in the support of $p_{\Theta_{h_P}}$ and $p_{\Theta_{h'_P}}$,

$$\frac{p_{\Theta_{h_P}}(\theta')}{p_{\Theta_{h_P}}(\theta)} = \frac{p_{\Theta_{h'_P}}(\theta')}{p_{\Theta_{h'_P}}(\theta)}.$$

The second condition implies non-decreasing hazard rates (see Lemma 8 in the appendix), a standard condition typically assumed in screening problems and other problems of information economics (Mussa and Rosen, 1978; Spence, 1980; Baron and Myerson, 1982; Maskin and Riley, 1984; Matthews and Moore, 1986; Bagnoli and Bergstorm, 2005). Monotone hazard rates facilitate solving for the optimal menu of contracts subject to the agent’s incentive constraints. Note that we weaken log-concavity to apply only to non-zero probabilities, which means that the marginal beliefs of the principal on marginal cost types are unimodal with respect to all marginal cost types that get assigned strictly positive probability.

The third condition says that, after becoming aware of additional types in $\Theta_{h'_P}$, the relative likelihood of types that she has been aware of at Θ_{h_P} should remain the same, provided that they are not ruled out. Such a condition has been suggested and axiomatized by Karni and Vierø (2013) for updating beliefs of a single decision maker upon becoming aware (see also Dominiak and Tserenjigmid, 2018).³ This condition is less compelling in a game theoretic setting in which, conditional on an information set, a player may not only increase their own awareness but at the same time infer from the opponent’s actions some information about the latter’s types as well. However, note that the assumption is mute for types that are assigned zero probability upon becoming aware (because, for instance, they can be ruled out from raising the principal’s awareness). This condition allows us to link first-level beliefs and relate menus of contracts across trees, and also facilitates the analysis of rational disclosure decisions by the agent.

In addition to the restrictions discussed above, we sometimes also impose a tie-breaking condition and common belief of this tie-breaking condition. We discuss the nature of this condition as well as its importance for some of our results later on in the text.

For any player $i = A, P$, a strategy s_i is *sequentially rational* at information set h_i with belief $\beta_i(h_i)$ if either s_i does not reach h_i or there does not exist a strategy s'_i that coincides with s_i for all information sets of player i preceding h_i and yields a higher expected payoff given $\beta_i(h_i)$.

³Their axioms feature invariant risk preferences; i.e., risk preferences that do not change with changes in awareness. This is also an implicit assumption made in games with unawareness. The assignment of payoffs to terminal histories does not depend on the game tree in the forest but just on the terminal history as long as the terminal history exists in the tree.

Let B_i denote player i 's set of belief systems. We apply a cautious version of extensive-form rationalizability with first-order marginal beliefs on types restricted by logconcavity and reverse Bayesianism.

Definition 1 (Δ -Prudent Rationalizable Strategies) For each player $i \in \{A, P\}$, define inductively the following sequences of belief systems and strategies:

$$\begin{aligned}
R_i^0 &= S_i \text{ and, for } k \geq 1, \\
B_A^k &= \left\{ \beta_A \in B_A : \begin{array}{l} \text{For every } h_A, \text{ the support of } \beta_A(h_A) \text{ is given by the set of all} \\ s_P \in R_P^{k-1, T_{h_A}} \text{ that reach } h_A, \text{ provided this set is non-empty.} \end{array} \right\} \\
B_P^k &= \left\{ \beta_P \in B_P : \begin{array}{l} \text{For every information set } h_P, \text{ the support of } \beta_P(h_P) \text{ is the} \\ \text{set of all } (s_A, \theta) \in R_A^{k-1, T_{h_P}} \times \Theta_{h_P} \text{ that reach } h_P \text{ if this set} \\ \text{is nonempty.} \end{array} \right\} \\
R_i^k &= \left\{ s_i \in R_i^{k-1} : \begin{array}{l} \text{There exists a belief system } \beta_i \in B_i^k \text{ with which, for every} \\ \text{information set } h_i \in H_i, s_i \text{ is sequentially rational at } h_i. \end{array} \right\}
\end{aligned}$$

The set of player i 's Δ -prudent rationalizable strategies is:⁴

$$R_i^\infty = \bigcap_{k=1}^{\infty} R_i^k.$$

This solution concept is easier to interpret than standard equilibrium solution concepts like perfect Bayesian equilibrium or sequential equilibrium. It captures common cautious (strong) belief in rationality (see Battigalli and Siniscalchi, 2002; Guarino, 2020) and common belief in reverse Bayesianism and in logconcavity of marginals on types. Important for our purpose, it does not assume that players are automatically certain of a ready-made convention of play upon becoming aware, like any equilibrium concept assumes, as it is not clear how this could be justified. Note that since it is an iterated elimination procedure on strategies, it yields predictions for every finite level k of mutual (strong) belief. This is akin to level- k reasoning in experimental game theory and should turn out useful for future experimental tests of our theory.⁵

Our solution concept combines features of the solution concepts of extensive-form rationalizability (Pearce, 1984; Battigalli, 1997), Δ -rationalizability or extensive-form best response sets (Battigalli and Siniscalchi, 2003; Battigalli and Friedenberg, 2012; Battigalli and Prestipino, 2013), and iterated admissibility and self-admissible sets (Brandenburger, Friedenberg, and

⁴We call our solution concept Δ -prudent rationalizability because it features restrictions on first-order beliefs. Rationalizability with such restrictions has been called Δ -rationalizability by Battigalli and Siniscalchi (2003) and Battigalli and Prestipino (2013), and extensive-form best response sets and directed rationalizability by Battigalli and Friedenberg (2012). Heifetz, Meier, and Schipper (2020) use the name prudent rationalizability for a cautious version of extensive-form rationalizability in order to distinguish it from cautious rationalizability in Pearce (1984), who presents another but related solution concept. Our solution concept features both caution and further restrictions.

⁵See Li and Schipper (2020, 2018) for experimental tests of prudent rationalizability without restrictions in disclosure games.

Keisler, 2008; Brandenburger and Friedenberg, 2010). Meier and Schipper (2012) show the equivalence of prudent rationalizability to iterated elimination of conditional weakly dominated strategies and to a version of iterated admissibility conditional on partial-games of the extensive-form game with unawareness.

3 Unawareness of Low Marginal Cost Events Only

3.1 Three Marginal Cost Types

Before we discuss a more general model, consider for simplicity two kinds of events that can affect marginal costs. One kind is reflected in the value $\phi \in \Phi = \{\underline{\phi}, \bar{\phi}\}$; the other, in the value $\psi \in \Psi = \{\underline{\psi}, \bar{\psi}\}$. Initially the principal is only aware of events of the first kind, Φ , while the agent is aware of both kinds of events. Total marginal costs are $\theta = \phi + \psi$ for $\phi \in \Phi$ and $\psi \in \Psi$.

Let us assume in this simple example that:

- (a) $\bar{\phi} > \underline{\phi} > 0$ and $\bar{\psi} > \underline{\psi}$,
- (b) $-\underline{\phi} < \underline{\psi} < 0 = \bar{\psi} < \bar{\phi}$, and
- (c) $\bar{\phi} - \underline{\phi} = \bar{\psi} - \underline{\psi} =: \Delta$.

With these assumptions, we have three distinct marginal cost types: The low marginal cost type $\underline{\phi} + \underline{\psi}$, the intermediate marginal cost type $\bar{\phi} + \underline{\psi} = \underline{\phi} + \bar{\psi} = \bar{\phi}$, and the high marginal cost type $\bar{\phi} + \bar{\psi} = \bar{\phi}$. Since the principal is aware of Φ but unaware of Ψ , she is unaware of the type with the lowest marginal cost, $\underline{\phi} + \underline{\psi}$. Thus, there are two trees: one in which nature selects $\underline{\phi}$ or $\bar{\phi}$ only, call it T_Φ , and one in which nature selects moves in $\Phi \times \Psi$, denoted by $T_{\Phi \times \Psi}$. Initially, the principal “lives” in tree T_Φ . Yet, the agent is aware of $\Phi \times \Psi$, so he can decide whether or not to make the principal aware of Ψ .

We characterize Δ -prudent rationalizable strategies level-by-level. At level 1, since the principal has full support beliefs, she offers a menu of exactly two contracts in the lower tree and a menu of exactly three contracts in the upper tree. Each menu has to maximize her expected profit given some full support belief over marginal cost types and strategies of the agent. The difference is that at the lower tree she is aware of two types only while at her information set in the upper tree she can take into account all three marginal cost types. The fact that in the upper tree she has been made aware of Ψ by the agent does not allow her to exclude any agent type because at level 1 no restrictions are implied yet on the agent’s strategies. In the lower tree, she can offer any optimal menu of two contracts in $([0, b]^2)^2$ with the belief that puts sufficiently large probability to one type of the agent accepting only the one contract and rejecting all others, and the other type accepting only the other contract and rejecting all others. Analogously, in the upper tree, she can offer any optimal menu of three contracts.

The agent at level 1, when faced by a menu of contracts in the last information set in any tree, selects a contract that maximizes his payoff unless none satisfies his participation constraint, in which case he selects the outside option. As shown in the appendix, the agent’s payoff function satisfies decreasing differences in quantities and marginal cost types (Lemma 2),

hence his optimal contract quantity is decreasing in his type (Lemma 3). Note that each of the last information sets of the agent in both trees are singleton. Thus, he knows the menu of contracts offered.

At his first information set in the upmost tree, any action can be optimal with an appropriate belief over the principal's contracts. He may make the principal aware of Ψ if he believes with sufficiently high probability that he gets a better deal. He may also keep the principal in the dark about Ψ if he believes with sufficiently high probability that it would result in a worse deal. This is because no restrictions on the principal's strategies can be assumed yet at level 1.

At level 2, the principal is certain of first-level Δ -prudent rationalizable strategies of the agent. Thus, she is certain that the agent observes participation constraints, self-selects to a contract in the menu according to his incentives, and that chosen quantities are monotone in the agent's type. Since level 1 imposes no restrictions on the agent's decision w.r.t. disclosure of Ψ , both disclosing and not are rationalizable for every type of the agent at level 1. Any belief system $\beta_P \in B_P^2$ for the principal must put strictly positive probability on any type making her aware when reaching her information set in the upper tree $T_{\Phi \times \Psi}$. Our solution concept requires in this case that her marginal beliefs on $\Phi \times \Psi$ satisfy reverse Bayesianism.

With such beliefs, any second-level Δ -prudent menu of contracts must satisfy the following. Let $\kappa^{(i)}$ be the order statistics of marginal costs $\phi + \psi$, with $\kappa^{(1)}$ being the high marginal cost type. Let p_Φ and $p_{\Phi \times \Psi}$ denote her marginal beliefs over marginal costs types at the lower and upper tree, respectively; e.g., p_Φ^i denotes her marginal probability for type $\kappa^{(i)}$ in tree T_Φ . Note that in tree T_Φ the principal is aware of $\kappa^{(1)}$ and $\kappa^{(2)}$ only. The principal's menu of contracts offered in tree $T_{\Phi \times \Psi}$ solves:

$$\max_{(q_{\Phi \times \Psi}^i, t_{\Phi \times \Psi}^i)_{i=1,2,3}} \sum_{i=1}^3 p_{\Phi \times \Psi}^i (v(q_{\Phi \times \Psi}^i) - t_{\Phi \times \Psi}^i)$$

subject to $IC_{\Phi \times \Psi}^{(i)}$: For $i = 2, 3$,

$$t_{\Phi \times \Psi}^i - \kappa^{(i)} q_{\Phi \times \Psi}^i \geq t_{\Phi \times \Psi}^{i-1} - \kappa^{(i)} q_{\Phi \times \Psi}^{i-1}$$

and $PC_{\Phi \times \Psi}^{(1)}$:

$$t_{\Phi \times \Psi}^1 - \kappa^{(1)} q_{\Phi \times \Psi}^1 \geq 0,$$

while the principal's menu of contracts in the lower tree T_Φ solves:

$$\max_{(q_\Phi^i, t_\Phi^i)_{i=1,2}} p_\Phi^1 (v(q_\Phi^1) - t_\Phi^1) + p_\Phi^2 (v(q_\Phi^2) - t_\Phi^2)$$

subject to $IC_\Phi^{(2)}$:

$$t_\Phi^2 - \kappa^{(2)} q_\Phi^2 \geq t_\Phi^1 - \kappa^{(2)} q_\Phi^1$$

and $PC_\Phi^{(1)}$:

$$t_\Phi^1 - \kappa^{(1)} q_\Phi^1 \geq 0.$$

In stating these problems, we make use of the fact that the principal is certain that the agent is rational, hence the incentive compatibility and participation constraints. We also make use of the fact that, as the principal's marginal on cost types is logconcave, solutions will satisfy a

monotonicity constraint: Higher marginal cost types select lower quantities (see Appendix A). With such a monotonicity constraint, only local upward (in terms of marginal costs) incentive compatibility constraints need to be considered (see Lemmas 4 and 5 in the appendix).

The solutions to the principal's programs are characterized as follows (see Appendix A): For the principal's information set in the upper tree $T_{\Phi \times \Psi}$, optimal outputs have to satisfy, for $i = 1, 2, 3$,

$$v'(\hat{q}_{\Phi \times \Psi}^i) = \kappa^{(i)} + \frac{\sum_{j>i} p_{\Phi \times \Psi}^j}{p_{\Phi \times \Psi}^i} (\kappa^{(i)} - \kappa^{(i+1)}),$$

and transfers are given by:

$$\hat{t}_{\Phi \times \Psi}^i = \kappa^{(i)} \hat{q}_{\Phi \times \Psi}^i + \sum_{j<i} (\kappa^{(j)} - \kappa^{(j+1)}) \hat{q}_{\Phi \times \Psi}^j.$$

For the principal's information set in the lower tree T_{Φ} , the optimal outputs satisfy:

$$v'(\hat{q}_{\Phi}^2) = \kappa^{(2)} = \underline{\phi}$$

$$v'(\hat{q}_{\Phi}^1) = \kappa^{(1)} + \frac{p_{\Phi}^2}{p_{\Phi}^1} (\kappa^{(1)} - \kappa^{(2)}),$$

and transfers are:

$$\hat{t}_{\Phi}^2 = \kappa^{(2)} \hat{q}_{\Phi}^2 + (\kappa^{(1)} - \kappa^{(2)}) \hat{q}_{\Phi}^1$$

and

$$\hat{t}_{\Phi}^1 = \kappa^{(1)} \hat{q}_{\Phi}^1.$$

Moreover, as the principal's beliefs are assumed to satisfy reverse Bayesianism, we can conclude that for $i = 1, 2$ (see Lemma 12 (i) in the appendix),

$$\hat{q}_{\Phi \times \Psi}^i < \hat{q}_{\Phi}^i.$$

To see this, for instance for $i = 1$, note that it suffices to show (since $v''(q) < 0$) that:

$$\begin{aligned} v'(\hat{q}_{\Phi \times \Psi}^1) &> v'(\hat{q}_{\Phi}^1) \\ \kappa^{(1)} + \frac{p_{\Phi \times \Psi}^2 + p_{\Phi \times \Psi}^3}{p_{\Phi \times \Psi}^1} (\kappa^{(1)} - \kappa^{(2)}) &> \kappa^{(1)} + \frac{p_{\Phi}^2}{p_{\Phi}^1} (\kappa^{(1)} - \kappa^{(2)}) \\ \frac{p_{\Phi \times \Psi}^2 + p_{\Phi \times \Psi}^3}{p_{\Phi \times \Psi}^1} &> \frac{p_{\Phi}^2}{p_{\Phi}^1} \\ \frac{p_{\Phi \times \Psi}^2}{p_{\Phi \times \Psi}^1} + \frac{p_{\Phi \times \Psi}^3}{p_{\Phi \times \Psi}^1} &> \frac{p_{\Phi}^2}{p_{\Phi}^1} \\ \frac{p_{\Phi}^2}{p_{\Phi}^1} + \frac{p_{\Phi \times \Psi}^3}{p_{\Phi \times \Psi}^1} &> \frac{p_{\Phi}^2}{p_{\Phi}^1}, \end{aligned}$$

where the last line follows from reverse Bayesianism.

For the agent, any level-1 rationalizable strategy is also level-2 rationalizable. At level 3, the agent is certain of level-2 rationalizable strategies of the principal. It should be obvious

that the highest marginal cost type, $\kappa^{(1)}$, is indifferent between disclosing Ψ or not. If he finds the principal offering a contract acceptable to him, then his participation constraint is binding. Otherwise, he takes the outside option. This holds no matter whether or not he discloses Ψ , because after disclosing Ψ he remains the type with the highest marginal cost and hence he is held to his outside option. Since he is indifferent between disclosing or not, we will use as a tie-breaking assumption that type $\kappa^{(1)}$ does *not* disclose, in order to facilitate the analysis. Note that because he is always the type with the highest marginal cost, he has nothing to gain from disclosure. On the other hand, disclosure may bear a cost, even an infinitesimal one. This motivates this tie-breaking assumption. We discuss the role of this assumption in our results further in the next subsection.

For the intermediate marginal cost type, $\kappa^{(2)}$, the disclosure decision is less trivial. He is certain of level-2 Δ -prudent rationalizable strategies of the principal. Thus, with any level-3 belief system, he is certain that the principal's menu of contracts offered upon becoming aware satisfies the monotonicity property of quantities for each type across trees implied by reverse Bayesianism. We claim that, with any such a belief system, $\kappa^{(2)}$ does not prefer to raise the principal's awareness. To prove our claim, we compare the payoffs of $\kappa^{(2)}$ with and without raising awareness. We observe that $\kappa^{(2)}$ strictly prefers not to disclose Ψ because:

$$\begin{aligned} u_A(\hat{q}_{\Phi \times \Psi}^2, \hat{t}_{\Phi \times \Psi}^2, \kappa^{(2)}) &< u_A(\hat{q}_{\Phi}^2, \hat{t}_{\Phi}^2, \kappa^{(2)}) \\ \hat{t}_{\Phi \times \Psi}^2 - \kappa^{(2)} \hat{q}_{\Phi \times \Psi}^2 &< \hat{t}_{\Phi}^2 - \kappa^{(2)} \hat{q}_{\Phi}^2 \\ \hat{t}_{\Phi \times \Psi}^1 - \kappa^{(2)} \hat{q}_{\Phi \times \Psi}^1 &< \hat{t}_{\Phi}^1 - \kappa^{(2)} \hat{q}_{\Phi}^1 \\ \kappa^{(1)} \hat{q}_{\Phi \times \Psi}^1 - \kappa^{(2)} \hat{q}_{\Phi \times \Psi}^1 &< \kappa^{(1)} \hat{q}_{\Phi}^1 - \kappa^{(2)} \hat{q}_{\Phi}^1 \\ (\kappa^{(1)} - \kappa^{(2)}) \hat{q}_{\Phi \times \Psi}^1 &< (\kappa^{(1)} - \kappa^{(2)}) \hat{q}_{\Phi}^1, \end{aligned}$$

which follows from our earlier observation that $\hat{q}_{\Phi \times \Psi}^1 < \hat{q}_{\Phi}^1$, an implication of reverse Bayesianism (Lemma 12 (i)). The third line follows from incentive compatibility; the fourth line follows from the participation constraint of marginal cost type $\kappa^{(1)}$.

Similar to type $\kappa^{(2)}$, the low cost type, $\kappa^{(3)}$, of which the principal is initially unaware, does not want to raise the principal's awareness. To see this, note that at level 3, he is certain of level-2 Δ -prudent rationalizable strategies of the principal. Thus, he is certain that the principal's menu of contracts offered upon becoming aware satisfies the monotonicity property of quantities for each type across trees implied by reverse Bayesianism of the principal. With any such belief system, $\kappa^{(3)}$ strictly prefers not to disclose Ψ :

$$\begin{aligned} u_A(\hat{q}_{\Phi \times \Psi}^3, \hat{t}_{\Phi \times \Psi}^3, \kappa^{(3)}) &< u_A(\hat{q}_{\Phi}^2, \hat{t}_{\Phi}^2, \kappa^{(3)}) \\ \hat{t}_{\Phi \times \Psi}^3 - \kappa^{(3)} \hat{q}_{\Phi \times \Psi}^3 &< \hat{t}_{\Phi}^2 - \kappa^{(3)} \hat{q}_{\Phi}^2 \\ \hat{t}_{\Phi \times \Psi}^2 - \kappa^{(3)} \hat{q}_{\Phi \times \Psi}^2 &< \hat{t}_{\Phi}^2 - \kappa^{(3)} \hat{q}_{\Phi}^2 \\ \hat{t}_{\Phi \times \Psi}^2 - \kappa^{(2)} \hat{q}_{\Phi \times \Psi}^2 + \kappa^{(2)} \hat{q}_{\Phi \times \Psi}^2 - \kappa^{(3)} \hat{q}_{\Phi \times \Psi}^2 &< \hat{t}_{\Phi}^2 - \kappa^{(2)} \hat{q}_{\Phi}^2 + \kappa^{(2)} \hat{q}_{\Phi}^2 - \kappa^{(3)} \hat{q}_{\Phi}^2 \\ u_A(\hat{q}_{\Phi \times \Psi}^2, \hat{t}_{\Phi \times \Psi}^2, \kappa^{(2)}) + (\kappa^{(2)} - \kappa^{(3)}) \hat{q}_{\Phi \times \Psi}^2 &< u_A(\hat{q}_{\Phi}^2, \hat{t}_{\Phi}^2, \kappa^{(2)}) + (\kappa^{(2)} - \kappa^{(3)}) \hat{q}_{\Phi}^2 \end{aligned}$$

We have already seen above that $u_A(\hat{q}_{\Phi \times \Psi}^2, \hat{t}_{\Phi \times \Psi}^2, \kappa^{(2)}) < u_A(\hat{q}_{\Phi}^2, \hat{t}_{\Phi}^2, \kappa^{(2)})$ in this case. Moreover, reverse Bayesianism implies $\hat{q}_{\Phi \times \Psi}^2 < \hat{q}_{\Phi}^2$ here (see Lemma 12 (i) in the appendix). So, the last inequality follows.

For the principal, all level-2 rationalizable strategies are also level-3 rationalizable since there is no change of strategies of the agent at level 2. Consequently, all level-3 rationalizable

strategies of the agent are also level-4 rationalizable. Finally, since none of the types raises the principal's awareness at level 3, she must believe in level-2 prudent rationalizable strategies of the agent upon becoming aware and no further reduction of her strategy set occurs at level 4. This concludes the analysis.

We observe that if the principal is unaware of low marginal cost events only, then in any Δ -prudent rationalizable outcome the principal offers a menu of contracts for all types of the agent of which she is aware. Moreover, none of the agent's types raises her awareness. Thus, in any Δ -prudent rationalizable outcome, the principal remains unaware of the low marginal cost events. There is bunching at the top as the low marginal cost type selects the same contract as the intermediate marginal cost type.

3.2 Any Finite Number of Marginal Cost Types

We now consider more generally any finite number of marginal cost types. It is not clear how general the observations of the prior example are. Is it still the case that there is no disclosure whatsoever when more types are allowed? With more types, there may be now also intermediate types of which the principal is initially unaware. Moreover, the question for the agent is now not just whether or not to raise the principal's awareness, but if yes by how much. We can address such questions in a model with more than three marginal cost types.

Suppose the cost function is defined by $c(\theta) = \max \bar{\Theta} + 1 - \theta$ for any $\theta \in \bar{\Theta}$. This marginal cost function is useful for discussing changes of awareness that do not change the types with the highest marginal costs. Spaces of moves of nature just differ by lower marginal cost types. When the principal's awareness is raised from Θ_P to $\Theta \supsetneq \Theta_P$, she faces all marginal cost types in Θ_P and some types with successively lower marginal costs in $\Theta \setminus \Theta_P$. We call the general model with any finite number of marginal cost types the model with unawareness of low marginal cost events only.

When a type of the agent is indifferent between raising the principal's awareness or not, we assume that he does not. This pertains especially to the type with the highest marginal cost. This type is indifferent between raising the principal's awareness or not because he is always held to his outside option. If communicating with the principal carries an infinitesimal cost, then this type would never want to raise the principal's awareness. Although we do not model costs of communication explicitly, we use this idea to break ties and assume that in such a case the type of the agent does not raise the principal's awareness. We discuss the crucial role of this assumption after stating the result.

Proposition 1 *Consider the model in which the principal is unaware of low marginal cost events only. Assume that it is commonly believed upon becoming aware that the highest marginal cost type does not raise the principal's awareness. In any Δ -prudent rationalizable outcome, the following holds:*

- (0) *The principal does not become aware of all of the agent's types.*
- (i) *None of the agent's types of which the principal is initially aware raises her awareness of further types.*

- (ii) *None of the agent's types of which the principal is initially unaware raises her awareness to a level that includes his own type.*
- (iii) *The principal offers an optimal menu of contracts for all types of which she has been or became aware.*
- (iv) *There is bunching at the top: Low marginal cost types of whom the principal remains unaware select a contract for the lowest marginal cost type of which she becomes aware.*

The proof is contained in the appendix. The intuition is as follows: The type with the highest marginal cost is indifferent between raising her awareness or not, because he is held to his outside option in any event. According to the tie-breaking assumption, he does not raise the principal's awareness. Raising the awareness of the principal of lower marginal costs reduces optimal quantities for types of which she has been previously aware. Since information rents earned by these types are weighted by those quantities, it would decrease the payoffs of types she conceives. Any type she is not aware of prefers her to be aware of all types with higher marginal costs but his rather than being aware of types with even lower marginal costs. Thus, if the principal's awareness is raised, she realizes that no type she is now aware of would want to raise her awareness to that extent. She cannot rationalize the agent's disclosure action any further and hence resorts to her lower-level strategy of offering a menu of contracts for all types she is aware that is optimal w.r.t. some of her belief systems.

We discuss the role of assumptions embodied in our solution concept. Assuming logconcavity of beliefs rules out (standard) bunching (see Appendix A). It ensures that the quantities offered to lower marginal cost types are larger, because it implies monotone inverse hazard rates in the principal's first-order conditions. It aligns incentives of the principal with incentives of the agent as the agent's payoff function has decreasing differences in quantities and marginal cost types. It also ensures that only local incentive compatibility constraints need to be considered and thus simplifies the principal's optimization problem. Without this assumption, the menu of contracts offered by the principal would be much more difficult to analyze.

Similarly, assuming reverse Bayesianism facilitates the comparison of the agent's payoff from making the principal aware or not. It says that the distribution over agent's types upon becoming aware respects the same relative likelihoods for types the principal has been aware of initially, unless they are ruled out. Therefore, it has implications for the inverse hazard rates in the principal's first-order conditions before and after becoming aware. Without this assumption, the agent's disclosure decision would be much more difficult to analyze and would depend on additional assumptions on the value function v .

Assuming that the principal knows that the type with the highest marginal costs would not raise her awareness facilitates the analysis and allows us to derive a nonempty set of strategies with our solution concept. Without this assumption, upon seeing disclosure, the principal may now be certain that only the type with the highest marginal cost made her aware. However, when offering a menu of a single contract that is tailor-made only for this type and hence ignores all incentive compatibility constraints for other types, all other types will want to raise her awareness as well. But when believing this and offering a menu of contracts for all those types, those types do not all want to disclose. Thus, we are led to a cycle. Here, our solution procedure does not lead to a reduction of the strategy sets and lacks consistency. In

other words, common belief in cautious rationality, logconcavity, and reverse Bayesian may be empty without the assumption of common belief in the tie-breaking condition.⁶ While such a restriction is foreign to rationalizability concepts, which place restrictions on beliefs rather than directly on behavior, they are common for equilibrium notions. When constructing mixed equilibria, the players are typically indifferent among an infinite number of mixtures and the game theorist picks the one that suits the construction. Here, we follow such a convention just for one type of which the principal is initially aware.

4 Unawareness of High Marginal Cost Events Only

4.1 Three Marginal Cost Types

Once again, before discussing a more general model, we consider for simplicity two kinds of events that can affect marginal costs: $\Phi = \{\underline{\phi}, \bar{\phi}\}$ and $\Psi = \{\underline{\psi}, \bar{\psi}\}$; initially the principal is only aware of Φ , while the agent is aware of both Φ and Ψ . Total marginal cost are $\theta = \phi + \psi$ for $\phi \in \Phi$ and $\psi \in \Psi$.

Now, we assume that:

- (a) $\bar{\phi} > \underline{\phi} > 0$ and $\bar{\psi} > \underline{\psi}$,
- (b) $-\underline{\phi} < \underline{\psi} = 0 < \bar{\psi} < \bar{\phi}$, and
- (c) $\bar{\phi} - \underline{\phi} = \bar{\psi} - \underline{\psi} =: \Delta$.

Assumptions (a) and (c) are identical to the ones in Section 3.1. Assumption (b) now states that $\underline{\psi} = 0 < \bar{\psi}$. This changes the interpretation of the model considerably. As before, there are three distinct marginal cost types; but now, the low marginal cost type is $\underline{\phi} + \underline{\psi} = \underline{\phi}$, the intermediate marginal cost type $\bar{\phi} + \underline{\psi} = \underline{\phi} + \bar{\psi}$ equals $\bar{\phi}$, and the high marginal cost type is $\bar{\phi} + \bar{\psi} > \bar{\phi}$. Since the principal is aware of Φ but unaware of Ψ , she is unaware of the type with the highest marginal costs, $\bar{\phi} + \bar{\psi}$, in contrast to Section 3.1.

There are still two trees in this setting. In the lower one, T_Φ , nature selects $\underline{\phi}$ or $\bar{\phi}$ only. In the upper one, $T_{\Phi \times \Psi}$, nature selects moves in $\Phi \times \Psi$. Initially the principal lives in tree T_Φ . Yet, the agent is aware of $\Phi \times \Psi$ and thus can decide whether or not to make the principal aware of Ψ .

We characterize Δ -prudent rationalizable strategies level-by-level. The analysis of level 1 is analogous to the analysis of level 1 in Section 3.1. The principal offers expected profit maximizing menus of two contracts in the lower tree and of at most three contracts in the upper tree given some full-support belief system. The agent at level 1 may or may not make the principal aware of Ψ and self-selects a contract from the menu according to his incentives. Since the agent's payoff has decreasing differences in quantities and marginal costs (Lemma 2),

⁶It is well known that further restrictions may not shrink the set of strategies but may yield a different set of strategies. See for an example on extensive-form rationalizability versus prudent rationalizability, Heifetz, Meier, and Schipper (2020, Section 4.2). Battigalli and Prestipino (2013) and Battigalli and Friedenber (2012) also discuss how restrictions affect Δ -rationalizability or extensive-form best response sets.

optimal contract quantities are decreasing in marginal cost type modulo transfer (Lemma 3). If his participation constraint cannot be satisfied with any contract, he selects the outside option.

At level 2, the principal is certain of first-level Δ -prudent rationalizable strategies of the agent. Thus, she is certain that the agent observes participation constraints and self-selects among contracts according to his incentives.

Since the first level imposes no restrictions on the agent's decision regarding the disclosure of Ψ , both actions are rationalizable for every type of the agent at level 1. So, any belief system $\beta_P \in B_P^2$ must put strict positive probability on any type making her aware when reaching her information set in the upper tree $T_{\Phi \times \Psi}$. Our solution concept requires in this case that her marginal belief on $\Phi \times \Psi$ satisfy reverse Bayesianism. Let $\kappa^{(i)}$ now be the dual order statistics of marginal costs $\phi + \psi$, with $\kappa^{(1)}$ being the low marginal cost type. Further, let p_Φ and $p_{\Phi \times \Psi}$ denote her marginal belief over marginal costs types in the lower and upper trees, respectively. Note that in tree T_Φ the principal is aware of $\kappa^{(1)}$ and $\kappa^{(2)}$ only. Then the principal's menu of contracts offered in tree $T_{\Phi \times \Psi}$ solves:

$$\max_{(q_{\Phi \times \Psi}^i, t_{\Phi \times \Psi}^i)_{i=1,2,3}} \sum_{i=1}^3 p_{\Phi \times \Psi}^i (v(q_{\Phi \times \Psi}^i) - t_{\Phi \times \Psi}^i)$$

subject to $\text{IC}_{\Phi \times \Psi}^{(i)}$: For $i = 1, 2$,

$$t_{\Phi \times \Psi}^i - \kappa^{(i)} q_{\Phi \times \Psi}^i \geq t_{\Phi \times \Psi}^{i+1} - \kappa^{(i)} q_{\Phi \times \Psi}^{i+1}$$

and $\text{PC}_{\Phi \times \Psi}^{(3)}$:

$$t_{\Phi \times \Psi}^3 - \kappa^{(3)} q_{\Phi \times \Psi}^3 \geq 0,$$

and the principal's menu of contracts in the lower tree T_Φ solves:

$$\max_{(q_\Phi^i, t_\Phi^i)_{i=1,2}} p_\Phi^1 (v(q_\Phi^1) - t_\Phi^1) + p_\Phi^2 (v(q_\Phi^2) - t_\Phi^2)$$

subject to $\text{IC}_\Phi^{(1)}$:

$$t_\Phi^1 - \kappa^{(1)} q_\Phi^1 \geq t_\Phi^2 - \kappa^{(1)} q_\Phi^2$$

and $\text{PC}_\Phi^{(2)}$:

$$t_\Phi^2 - \kappa^{(2)} q_\Phi^2 \geq 0.$$

Again, in stating these problems, we make use of the fact that the principal at level 2 is certain that the agent is rational and that the principal's belief on agent types is logconcave.

The solutions to the principal's programs are characterized as follows (further details in Appendix A): For the principal's information set in the upper tree, the optimal outputs have to satisfy for $i = 1, 2, 3$,

$$v'(\hat{q}_{\Phi \times \Psi}^i) = \kappa^{(i)} + \frac{\sum_{j < i} p_{\Phi \times \Psi}^j (\kappa^{(i)} - \kappa^{(i-1)})}{p_{\Phi \times \Psi}^i},$$

and transfers are given by:

$$\hat{t}_{\Phi \times \Psi}^i = \kappa^{(i)} \hat{q}_{\Phi \times \Psi}^i + \sum_{j > i} (\kappa^{(j)} - \kappa^{(j-1)}) \hat{q}_{\Phi \times \Psi}^j.$$

For the principal's information set in the lower tree the optimal outputs satisfy:

$$v'(\hat{q}_{\Phi}^1) = \kappa^{(1)} = \underline{\phi}$$

$$v'(\hat{q}_{\Phi}^2) = \kappa^{(2)} + \frac{p_{\Phi}^1}{p_{\Phi}^2}(\kappa^{(2)} - \kappa^{(1)}),$$

and transfers are:

$$\hat{t}_{\Phi}^1 = \kappa^{(1)}\hat{q}_{\Phi}^1 + (\kappa^{(2)} - \kappa^{(1)})\hat{q}_{\Phi}^2$$

and

$$\hat{t}_{\Phi}^2 = \kappa^{(2)}\hat{q}_{\Phi}^2.$$

The programs and the first-order conditions are as in Section 3, except that we use now the dual order statistics of marginal costs types.

Since the principal's marginal beliefs on types are assumed to satisfy reverse Bayesianism, we conclude (see Lemma 12 (ii) in the appendix) that for that for all $i = 1, 2$,

$$\hat{q}_{\Phi \times \Psi}^i = \hat{q}_{\Phi}^i.$$

To see this for $i = 2$, it suffices to show that:

$$\begin{aligned} v'(\hat{q}_{\Phi \times \Psi}^i) &= v'(\hat{q}_{\Phi}^i) \\ \kappa^{(2)} + \frac{p_{\Phi \times \Psi}^1}{p_{\Phi \times \Psi}^2}(\kappa^{(2)} - \kappa^{(1)}) &= \kappa^{(2)} + \frac{p_{\Phi}^1}{p_{\Phi}^2}(\kappa^{(2)} - \kappa^{(1)}) \\ \frac{p_{\Phi \times \Psi}^1}{p_{\Phi \times \Psi}^2} &= \frac{p_{\Phi}^1}{p_{\Phi}^2}, \end{aligned}$$

where the last line follows from reverse Bayesianism.

For the agent, any level-1 rationalizable strategy is also level-2 rationalizable. At level 3, the agent is certain of level-2 Δ -prudent rationalizable strategies of principal. It is obvious that the highest marginal cost type, $\kappa^{(3)}$, is indifferent between disclosing Ψ or not because after disclosing Ψ he would remain the highest marginal cost type and be held to his outside option.

The intermediate marginal cost type, $\kappa^{(2)}$, strictly prefers to disclose Ψ . To see this, note that without disclosure, he is held to the payoff of his outside option. With disclosure, he can earn positive information rents. More formally,

$$\begin{aligned} u_A(\hat{q}_{\Phi \times \Psi}^2, \hat{t}_{\Phi \times \Psi}^2, \kappa^{(2)}) &\geq u_A(\hat{q}_{\Phi}^2, \hat{t}_{\Phi}^2, \kappa^{(2)}) \\ \hat{t}_{\Phi \times \Psi}^2 - \kappa^{(2)}\hat{q}_{\Phi \times \Psi}^2 &\geq 0 \\ \hat{t}_{\Phi \times \Psi}^3 - \kappa^{(2)}\hat{q}_{\Phi \times \Psi}^3 &\geq 0 \\ \kappa^{(3)}\hat{q}_{\Phi \times \Psi}^3 - \kappa^{(2)}\hat{q}_{\Phi \times \Psi}^3 &\geq 0 \\ (\kappa^{(3)} - \kappa^{(2)})\hat{q}_{\Phi \times \Psi}^3 &\geq 0, \end{aligned}$$

where the third line follows from incentive compatibility and the fourth line from the highest marginal cost type's participation constraint.

The lowest marginal cost type, $\kappa^{(1)}$, also has an incentive to disclose Ψ . To see this note that:

$$\begin{aligned}
u_A(\hat{q}_{\Phi \times \Psi}^1, \hat{t}_{\Phi \times \Psi}^1, \kappa^{(1)}) &\geq u_A(\hat{q}_{\Phi}^1, \hat{t}_{\Phi}^1, \kappa^{(1)}) \\
\hat{t}_{\Phi \times \Psi}^1 - \kappa^{(1)} \hat{q}_{\Phi \times \Psi}^1 &\geq \hat{t}_{\Phi}^1 - \kappa^{(1)} \hat{q}_{\Phi}^1 \\
\hat{t}_{\Phi \times \Psi}^2 - \kappa^{(1)} \hat{q}_{\Phi \times \Psi}^2 &\geq \hat{t}_{\Phi}^2 - \kappa^{(1)} \hat{q}_{\Phi}^2 \\
\hat{t}_{\Phi \times \Psi}^2 - \kappa^{(2)} \hat{q}_{\Phi \times \Psi}^2 + \kappa^{(2)} \hat{q}_{\Phi \times \Psi}^2 - \kappa^{(1)} \hat{q}_{\Phi \times \Psi}^2 &\geq \hat{t}_{\Phi}^2 - \kappa^{(2)} \hat{q}_{\Phi}^2 + \kappa^{(2)} \hat{q}_{\Phi}^2 - \kappa^{(1)} \hat{q}_{\Phi}^2 \\
u_A(\hat{q}_{\Phi \times \Psi}^2, \hat{t}_{\Phi \times \Psi}^2, \kappa^{(2)}) + (\kappa^{(2)} - \kappa^{(1)}) \hat{q}_{\Phi \times \Psi}^2 &\geq u_A(\hat{q}_{\Phi}^2, \hat{t}_{\Phi}^2, \kappa^{(2)}) + (\kappa^{(2)} - \kappa^{(1)}) \hat{q}_{\Phi}^2 \\
u_A(\hat{q}_{\Phi \times \Psi}^2, \hat{t}_{\Phi \times \Psi}^2, \kappa^{(2)}) + (\kappa^{(2)} - \kappa^{(1)}) \hat{q}_{\Phi}^2 &\geq u_A(\hat{q}_{\Phi}^2, \hat{t}_{\Phi}^2, \kappa^{(2)}) + (\kappa^{(2)} - \kappa^{(1)}) \hat{q}_{\Phi}^2 \\
u_A(\hat{q}_{\Phi \times \Psi}^2, \hat{t}_{\Phi \times \Psi}^2, \kappa^{(2)}) &\geq u_A(\hat{q}_{\Phi}^2, \hat{t}_{\Phi}^2, \kappa^{(2)}).
\end{aligned}$$

The third line follows from incentive compatibility. The second last line, from the fact that quantities for types of whom the principal has been aware are invariant to changes of awareness, an implication from reverse Bayesianism (Lemma 12 (ii)). The last line follows from the previously proved inequality for the intermediate marginal cost type. We conclude that at level 3 all types of the agent have an incentive to raise the principal's awareness to $\Phi \times \Psi$.

For the principal, there is no reduction of the set of strategies a level 3 since there were none for the agent at level 2. Consequently, at level 4 there is no reduction of the set of strategies of the agent.

The level 4 principal is certain of level-3 Δ -prudent rationalizable strategies of the agent. Thus, when made aware, show knows any type could made have her aware. Thus, she continues to take all types into account. No further reduction of her strategies is possible.

This completes the analysis of the model with tree types in which the principal is unaware of high marginal cost types only. We observe that in any Δ -prudent rationalizable outcome, every type of the agent raises her awareness, except perhaps for the type with the highest marginal costs, who is indifferent. In any case, she offers a menu of contracts optimal for all types of which she has become aware.

4.2 Any Finite Number of Marginal Cost Types

We now consider any finite number of marginal cost types. Suppose the cost function is defined by $c(\theta) = \theta + 1$ for any $\theta \in \bar{\Theta}$. This marginal cost function is useful for discussing changes of awareness in which the lowest marginal cost types remain unchanged. When the principal's awareness is raised from Θ_P to $\Theta \supsetneq \Theta_P$, she faces all marginal cost types in Θ_P and some types with successively higher marginal costs in $\Theta \setminus \Theta_P$. We call this the general model in which the principal is unaware of high marginal cost events only.

Proposition 2 *Consider the model in which the principal is unaware of higher marginal cost events only. In any Δ -prudent rationalizable outcome, the following holds:*

- (i) *All types of the agent prefer to fully raise the principal's awareness except for the highest marginal cost type, who is indifferent.*

- (ii) *If it is common belief upon becoming aware that the highest marginal cost type would raise the principal's awareness, then any outcome involves full awareness.*
- (iii) *The principal offers an optimal menu of contracts of all types of which she is or becomes aware.*

The proof is contained in the appendix. On one hand, this result sounds intuitive. If there are events that could potentially increase marginal costs, why not make the principal aware of it? The type with the highest marginal costs is indifferent since he receives the utility of his outside option anyway. Other types benefit via additional information rents. Yet, the result is not obvious because types of whom the principal has already been aware benefit from higher information rents but are also harmed by the lower transfers to the newly-discovered higher marginal cost types. Our result verifies that last effect is overcompensated by additional information rents.

The analytical roles of the assumptions of common belief in rationality, caution, logconcavity, and reverse Bayesianism are the same as in Proposition 1. We do not make a tie-breaking assumption for the type with the highest marginal cost except for part (ii). This is because it is enough for the principal to have full support beliefs over strategies of the agent such that she does not rule out this type having raised her awareness upon becoming aware. Adopting the tie-breaking assumption of part (ii) for the entire analysis would not lead to any changes. Now, the tie-breaking assumption of part (ii) is different from the one used in Section 3: When a type of the agent is indifferent between raising the principal's awareness or not, he raises the principal's awareness, while in Section 3 we assumed he would not. Our tie-breaking assumption is perhaps more compelling now: If there is any uncertainty about how the principal reacts when becoming aware of a type, the highest type may entertain infinitesimal optimistic beliefs w.r.t. the contracts he will be offered upon disclosing. Although we do not model this explicitly here, we use this idea as motivation.

What would happen here if, instead, we made the tie-breaking assumption of Proposition 1? Suppose that the high cost type never raises the principal's awareness. Then, the principal would be certain of this at the next level of the solution procedure. Consequently, she would offer a menu of contracts optimal for the second-highest marginal cost type being the new highest marginal cost type, who is now held to his outside option. At the next level of the solution procedure, the second-highest marginal cost type would have no incentive to raise the principal's awareness. At first glance, it is easy to conjecture that, by induction, not raising awareness unravels and no type of the agent wants to raise the principal's awareness. A subtle issue is that a menu with contracts for less types is obviously not an element of the set of strategies with menus of contracts for all types. So this would not lead to a reduction of strategy sets. It is not clear to us how this "silence equilibrium" could be consistent with common cautious belief in rationality and the restrictions.

5 Discussion

The previous observations can be captured in a simple punch line: *The principal is happily made aware of inefficiencies but kept tacitly in the dark about some efficiencies.*

We interpret our model as one of changing the awareness level of the principal. How is it different from the principal assigning zero probability to some marginal cost types? If the principal assigns zero probability to some marginal cost type θ , then she assigns probability 1 to the complement of θ ; that is, she is certain that the agent is not of type θ . Hence, receiving a message like “Have you considered that the agent could have type θ ?” (a question) or “The agent could be of type θ or not of type θ .” (a tautology) does not contain inherent information—unless it gets information attributed strategically in a cheap-talk game. It is not clear why such a message should change the principal’s probabilistic assessment of θ , especially if she is absolutely certain that the agent’s type is not θ . Yet, if she were unaware of θ , she realizes upon receiving such a message that she did not consider that the agent’s type could be θ and hence may reevaluate her probabilistic assessment of the other types. Thus, we find the interpretation of unawareness much more compelling for the implications of belief change we study in this paper.

Whether or not the principal is unaware of θ or just assigns zero probability to it can be tested in a choice experiment (Schipper, 2013). Suppose that, besides contracting with the agent, the principal also contracts with some other party; e.g., an insurance contract, if she were risk averse. This contract could have a clause, say, in an addendum on p. 976 specifying that something bad happens if the agent is of type θ and something good if he is not. There could be also a second contract that is identical to the first except that in the addendum on p. 976 it is specified that no matter whether the agent is of type θ something good happens. If the principal is indifferent between the two contracts, this is consistent with her both assigning zero probability to θ or being unaware of θ (presumably because she did not bother to read the addendum). Now, consider a third contract identical to the first two except that in said addendum it specifies something good when the agent is of type θ and something bad when he is not. If the principal is indifferent between second and third contracts, then this is consistent with either assigning zero probability to “not θ ” or being unaware of θ ; but she cannot assign zero probability to both θ and its complement, so it must be that she is unaware of θ .

Probability zero may also lead to different implications in our model in which the principal is unaware of low marginal cost types only. In this model, at level 4 of the rationalizability procedure, the principal cannot rationalize if she finds herself at an information set in which she has become aware of additional marginal cost types. Consequently, she resorts to her level 3 (which are equivalent to her level 2) rationalizable contract menus. If instead the principal had assigned zero probability to some types, she could alternatively now give up this belief restriction and, very much in the spirit of the best rationalization principle embodied in rationalizability notions (Battigalli, 1996), perfectly rationalize the disclosure move of the agent.

This discussion begs the question whether the principal should not be aware that she is unaware of some types; in principle, we can always imagine that there are more or less efficient types. The difficulty in answering this question is perhaps better illustrated in a multidimensional screening problem. The principal could be aware that she is unaware of *some* dimension. The issue is that she cannot be aware that she is unaware of a *particular* dimension—otherwise, she would be aware of it; this is a standard feature of propositional unawareness (see Heifetz, Meier, and Schipper, 2006). She could be aware that she is unaware of *something*. The question, then, is what can she do about it? If she can investigate (e.g., consult with a specialist), then this should be modelled within the game (like in Heifetz, Meier, and Schipper, 2013; or Halpern and Rêgo, 2014). Moreover, in many settings, principals like regulators, CEOs etc. have to

justify their actions, which sometimes requires them to testify what exactly they had taken or not taken into account. Thus, we do find our focus of awareness of specific types relevant.

We agree that unawareness would be more compelling in a multidimensional setting in which the principal is unaware of some dimensions, rather than of more efficient or less efficient types in our one-dimensional setting. To some extent, we motivate this with our two-dimensional setting for our three-type models. We opted here for one-dimensional settings of the general finite models for simplicity, and consider the extensions to the general multi-dimensional case for future research.

We believe that there are various other avenues for future research: It would be interesting to combine disclosure of awareness with disclosure of information, the latter having recently been studied by Pram (2020) and Ali, Lewis, and Vasserman (2019). It would also be interesting to explore the consequences of giving up belief restrictions like logconcavity and reverse Bayesianism. Finally, the clear differences in the theoretical predictions of our two models lend themselves to experimental testing.

A Preliminaries

In this appendix we collect results that we repeatedly use throughout the analysis. Consider a finite number of marginal cost types with order statistics $\kappa^{(1)} > \kappa^{(2)} > \dots > \kappa^{(n)}$. Let p^i the principal's probability assigned to marginal cost type $\kappa^{(i)}$ for $i = 1, \dots, n$. Her constrained optimization problem given her awareness of types $\kappa^{(1)}, \dots, \kappa^{(n)}$ is:

$$\max_{(q^i, t^i)_{i=1, \dots, n} \in ([0, b]^2)^n} \sum_{i=1}^n p^i (v(q^i) - t^i)$$

subject to, for all $i = 1, \dots, n$ and $j = 1, \dots, n$:

IC $_{i,j}$:

$$t^i - k^{(i)} q^i \geq t^j - k^{(i)} q^j$$

PC $_i$:

$$t^i - k^{(i)} q^i \geq 0.$$

Lemma 1 For all $i = 2, \dots, n$, PC $_1$ and IC $_{i,i-1}$ implies PC $_i$.

PROOF. Observe that:

$$0 \leq t^1 - \kappa^{(1)} q^1 \stackrel{\kappa^{(1)} > \kappa^{(2)}}{\leq} t^1 - \kappa^{(2)} q^1 \stackrel{\text{IC}_{2,1}}{\leq} t^2 - \kappa^{(2)} q^2 \\ \stackrel{\kappa^{(2)} > \kappa^{(3)}}{\leq} t^2 - \kappa^{(3)} q^2 \leq \dots \stackrel{\text{IC}_{n,n-1}}{\leq} t^n - \kappa^{(n)} q^n.$$

This establishes the result. □

Recall that $u_A(t, q, \kappa) = t - \kappa q$.

Lemma 2 For every $t \in \mathbb{R}$, $u_A(t, q, \kappa)$ has strictly decreasing differences in (q, κ) : for any $q'' > q'$ and $\kappa'' > \kappa'$,

$$u_A(q'', t, \kappa'') - u_A(q'', t, \kappa') < u_A(q', t, \kappa'') - u_A(q', t, \kappa').$$

PROOF. For any $t \in \mathbb{R}$ and $q'' > q'$ and $\kappa'' > \kappa'$,

$$\begin{aligned} u_A(q'', t, \kappa'') - u_A(q'', t, \kappa') &< u_A(q', t, \kappa'') - u_A(q', t, \kappa') \\ t - \kappa'' q'' - t + \kappa' q'' &< t - \kappa'' q' - t + \kappa' q' \\ (\kappa' - \kappa'') q'' &< (\kappa' - \kappa'') q'. \end{aligned}$$

The result follows. □

The next observation follows from the previous lemmas and a generalization of Topkis' theorem by Edlin and Shannon (1998, Theorem 1).

Lemma 3 For every $t \in \mathbb{R}$, the agent's optimal q is strictly decreasing in κ .

Lemma 4 If for all $i = 2, \dots, n$, $q^i \geq q^{i-1}$ ($q^i > q^{i-1}$), then $IC_{i,i-1}$ implies $IC_{i,j}$ (with strict inequality) for all $j < i$.

PROOF. We prove by induction on the order statistics of marginal cost types.

Base case: For $i = 3, \dots, n$,

$$t^i - \kappa^{(i)} q^i \geq (>) t^{i-2} - \kappa^{(i)} q^{i-2}.$$

Rewrite $IC_{i-1,i-2}$:

$$\begin{aligned} t^{i-1} - \kappa^{(i-1)} q^{(i-1)} &\geq t^{i-2} - \kappa^{(i-1)} q^{i-2} \\ t^{i-1} - t^{i-2} &\geq \kappa^{(i-1)} (q^{i-1} - q^{i-2}). \end{aligned}$$

Since $q^{i-1} - q^{i-2} \geq (>) 0$ and $\kappa^{(i)} < \kappa^{(i-1)}$,

$$\begin{aligned} t^{i-1} - t^{i-2} &\geq (>) \kappa^{(i)} (q^{i-1} - q^{i-2}) \\ t^{i-1} - \kappa^{(i)} q^{i-1} &\geq (>) t^{i-2} - \kappa^{(i)} q^{i-2} \end{aligned}$$

implies now:

$$t^i - \kappa^{(i)} q^{(i)} \stackrel{IC_{i,i-1}}{\geq} t^{i-1} - \kappa^{(i)} q^{i-1} \geq (>) t^{i-2} - \kappa^{(i)} q^{i-2}.$$

This proves the base case.

Induction hypothesis: For $i, j = 2, \dots, n$ with $i > j$,

$$t^i - \kappa^{(i)} q^i \geq t^{i-j} - \kappa^{(i)} q^{i-j}.$$

Inductive step: For $i, j = 2, \dots, n$ with $j > j$, rewrite $IC_{i-j,i-j-1}$:

$$t^{i-j} - \kappa^{(i-j)} q^{(i-j)} \geq t^{i-j-1} - \kappa^{(i-j)} q^{i-j-1}$$

$$t^{i-j} - t^{i-j-1} \geq \kappa^{(i-j)}(q^{i-j} - q^{i-j-1}).$$

Since $q^{i-j} - q^{i-j-1} \geq (>)0$ and $\kappa^{(i)} < \kappa^{(i-j)}$,

$$\begin{aligned} t^{i-j} - t^{i-j-1} &\geq (>) \kappa^{(i)}(q^{i-j} - q^{i-j-1}) \\ t^{i-j} - \kappa^{(i)}q^{(i-j)} &\geq (>) t^{i-j-1} - \kappa^{(i)}q^{i-j-1} \end{aligned}$$

implies now:

$$t^i - \kappa^{(i)}q^i \stackrel{\text{Ind. Hyp.}}{\geq} t^{i-j} - \kappa^{(i)}q^{i-j} \geq (>) t^{i-j-1} - \kappa^{(i)}q^{i-j-1}.$$

This completes the proof. \square

Analogously, we can prove the following lemma:

Lemma 5 *If for all $i = 2, \dots, n$, $q^i \geq q^{i-1}$ ($q^i > q^{i-1}$), then $IC_{i,i+1}$ implies $IC_{i,j}$ (with strict inequality) for all j with $n \geq j > i$.*

Lemma 6 *For all $i = 2, \dots, n$, $IC_{i,i-1}$ bind in the principal's optimum.*

PROOF. We prove by contradiction. Suppose that for some $i = 2, \dots, n$, $IC_{i,i-1}$ does not bind:

$$t^i - \kappa^{(i)}q^i > t^{i-1} - \kappa^{(i)}q^{i-1}.$$

Then, the principal can decrease the transfer to marginal cost type $\kappa^{(i)}$ by $t^i - t^{i-1} - \kappa^{(i)}(q^i - q^{i-1}) > 0$. This increases her expected payoff while satisfying $IC_{i,i-1}$ with equality. Observe that $IC_{j,j-1}$ would be still satisfied for all $j = 1, \dots, n$. \square

Lemma 7 *For all $i = 1, \dots, n-1$, $IC_{i,i+1}$ are satisfied in the principal's optimum.*

PROOF. For any $i = 2, \dots, n$, Lemma 6 implies:

$$\begin{aligned} t^i - \kappa^{(i)}q^i &= t^i - \kappa^{(i)}q^{i-1} \\ t^i - t^{i-1} &= \kappa^{(i)}(q^i - q^{i-1}). \end{aligned}$$

Lemma 3 implies $q^i - q^{i-1} \geq 0$. Since $\kappa^{(i-1)} > \kappa^{(i)}$, we have:

$$\begin{aligned} t^i - t^{i-1} &\leq \kappa^{(i-1)}(q^i - q^{i-1}) \\ t^i - \kappa^{(i-1)}q^i &\leq t^{i-1} - \kappa^{(i-1)}q^{i-1}, \end{aligned}$$

as desired. \square

Remark 1 PC_1 is binding in the principal's optimum.

Thus, we can reduce the principal's optimization problem is reduced to:

$$\max_{(q^i, t^i)_{i=1, \dots, n} \in ([0, b]^2)^n} \sum_{i=1}^n p^i (v(q^i) - t^i)$$

subject to, for all $i = 2, \dots, n$,

IC_{*i,i-1*}:

$$t^i - \kappa^{(i)} q^i = t^{i-1} - \kappa^{(i)} q^{i-1}$$

PC₁:

$$t^1 - \kappa^{(1)} q^1 = 0$$

M_{*i,i-1*}:

$$q^i \geq q^{i-1}.$$

Note that, even though monotonicity is implied by decreasing differences of the agent's objective function (Lemmas 2 and 3), it is a constraint for the principal.

We can rewrite the first two classes of constraints recursively as, for $i = 2, \dots, n$,

$$t^i = \sum_{j=1}^{i-1} (\kappa^{(j)} - \kappa^{(j+1)}) q^j + \kappa^{(i)} q^i,$$

$$t^1 = \kappa^{(1)} q^1.$$

We omit momentarily the monotonicity constraints. (These constraints will be verified later.) Substituting the remaining constraints into the principal's objective function yields the unconstrained optimization problem:

$$\max_{(q^i)_{i=1, \dots, n} \in [0, b]^n} p^1 \left(v(q^1) - \kappa^{(1)} q^1 \right) + \sum_{i=2}^n p^i \left(v(q^i) - \sum_{j=1}^{i-1} (\kappa^{(j)} - \kappa^{(j+1)}) q^j - \kappa^{(i)} q^i \right).$$

Deriving first-order conditions yields: For $i = 1, \dots, n$,

$$v'(\hat{q}^i) = \kappa^{(i)} + \frac{\sum_{j>i}^n p^j}{p^i} (\kappa^{(i)} - \kappa^{(i+1)})$$

$$\hat{t}^i = \sum_{j=1}^{i-1} (\kappa^{(j)} - \kappa^{(j+1)}) \hat{q}^j + \kappa^{(i)} \hat{q}^i,$$

where $\frac{\sum_{j>i}^n p^j}{p^i}$ is the inverse hazard rate. Transfers can be rewritten recursively as:

$$\hat{t}^i = \hat{t}^1 + \sum_{1 < j \leq i} \kappa^{(j)} (\hat{q}^j - q^{j-1}),$$

which separates them into the transfer to the highest marginal-cost type and the information rents.

The lowest marginal cost type provides efficient output (“no distortion at the top”), i.e.,

$$v'(\hat{q}^n) = \kappa^{(n)},$$

while all other types underprovide in the principal's optimum. Monotonicity implies that lower marginal cost types receive higher transfers. However, monotonicity is not implied by the first-order conditions; we need an additional condition. Since $v'(q) < 0$, we have for all $i = 1, \dots, n-1$,

$$\hat{q}^i \leq \hat{q}^{i+1}$$

$$\begin{aligned}
v'(\hat{q}^i) &\geq v'(\hat{q}^{i+1}) \\
\kappa^{(i)} + \frac{\sum_{j>i}^n p^j}{p^i} (\kappa^{(i)} - \kappa^{(i+1)}) &\geq \kappa^{(i+1)} + \frac{\sum_{j>i+1}^n p^j}{p^{i+1}} (\kappa^{(i+1)} - \kappa^{(i+2)}) \\
\kappa^{(i)} - \kappa^{(i+1)} + \frac{\sum_{j>i}^n p^j}{p^i} (\kappa^{(i)} - \kappa^{(i+1)}) &\geq \frac{\sum_{j>i+1}^n p^j}{p^{i+1}} (\kappa^{(i+1)} - \kappa^{(i+2)}).
\end{aligned}$$

Assume equidistant marginal cost types:

Assumption 1 For all $i, j = 1, \dots, n-1$ such that $i+j+1 \leq n$, we have $\kappa^{(i)} - \kappa^{(i+1)} = \kappa^{(i+j)} - \kappa^{(i+j+1)}$.

Then, for all $i = 1, \dots, n-1$ previous inequality becomes:

$$1 + \frac{\sum_{j>i}^n p^j}{p^i} \geq \frac{\sum_{j>i+1}^n p^j}{p^{i+1}}.$$

A sufficient condition for previous inequality to hold is:

$$\frac{\sum_{j>i}^n p^j}{p^i} \geq \frac{\sum_{j>i+1}^n p^j}{p^{i+1}}.$$

That is, the inverse hazard rate is non-increasing in i (or the hazard rate is non-decreasing in i). A sufficient condition for this is logconcavity.

Definition 2 Probability distribution p is logconcave if for all $i = 2, \dots, n-1$,

$$p^i p^i \geq p^{i+1} p^{i-1}.$$

Lemma 8 If p is logconcave, then:

(i) Relative likelihoods are non-increasing in i : For any $i, j = 1, \dots, n$ and m such that $j > i$ and $j+m \leq n$,

$$\frac{p^{i+m}}{p^i} \geq \frac{p^{j+m}}{p^j}.$$

(ii) Inverse hazards rates are non-increasing in i : For any $i, j = 1, \dots, n$ with $j > i$,

$$\frac{\sum_{m>i}^n p^m}{p^i} \geq \frac{\sum_{m>j}^n p^m}{p^j}.$$

PROOF. Let p be logconcave and $p^i > 0$ for all $i = 1, \dots, n$. Note that for $i = 1, \dots, n-2$,

$$\begin{aligned}
p^{i+1} p^{i+1} &\geq p^i p^{i+2} \\
\frac{p^{i+1}}{p^i} &\geq \frac{p^{i+2}}{p^{i+1}}
\end{aligned}$$

$$\log p^{i+1} - \log p^i \geq \log p^{i+2} - \log p^{i+1}.$$

Inductively, we have for $j > i$, $j + 1 \leq n$,

$$\log p^{i+1} - \log p^i \geq \log p^{j+1} - \log p^{j+1}.$$

(i) For any $i, j = 1, \dots, n$ and m such that $j + m \leq n$,

$$\begin{aligned} \frac{p^{i+m}}{p^i} &= \frac{p^{i+1}}{p^i} \frac{p^{i+2}}{p^{i+1}} \cdots \frac{p^{i+m}}{p^{i+m-1}} \\ \log \left(\frac{p^{i+m}}{p^i} \right) &= \log \left(\frac{p^{i+1}}{p^i} \frac{p^{i+2}}{p^{i+1}} \cdots \frac{p^{i+m}}{p^{i+m-1}} \right) \\ &= (\log p^{i+1} - \log p^i) + (\log p^{i+2} - \log p^{i+1}) + \dots + (\log p^{i+m} - \log p^{i+m-1}) \\ &\geq (\log p^{j+1} - \log p^j) + (\log p^{j+2} - \log p^{j+1}) + \dots + (\log p^{j+m} - \log p^{j+m-1}) \\ &= \log \left(\frac{p^{j+1}}{p^j} \frac{p^{j+2}}{p^{j+1}} \cdots \frac{p^{j+m}}{p^{j+m-1}} \right) = \log \left(\frac{p^{j+m}}{p^j} \right) \\ &= \frac{p^{i+1}}{p^i} \frac{p^{i+2}}{p^{i+1}} \cdots \frac{p^{i+m}}{p^{i+m-1}} = \frac{p^{i+m}}{p^i}, \end{aligned}$$

where the inequality follows from log concavity, i.e., above inequality, applied to each term of the sum.

(ii) Rewrite (i) as:

$$p^j p^{i+m} - p^i p^{j+m} \geq 0.$$

Then,

$$\begin{aligned} \sum_{m=1}^{n-j} (p^j p^{i+m} - p^i p^{j+m}) &\geq 0 \\ p^j \left(\sum_{m>i}^{n-j+i} p^m \right) - p^i \left(\sum_{m>j}^n p^m \right) &\geq 0 \\ p^j \left(\sum_{m>i}^n p^m \right) - p^i \left(\sum_{m>j}^n p^m \right) &\geq 0. \end{aligned}$$

This establishes the lemma. □

We conclude from Lemma 8:

Lemma 9 *If p is logconcave, the first-order conditions above are valid for the solution to principal's optimization problem.*

While the previous observations should be well-known, we were unable to locate a complete treatment of the finite case in the literature.

The next lemma establishes that reverse Bayesian updating is consistent with logconcavity if the update is a truncation.

Lemma 10 Let $\Theta'' = \{1, \dots, n\}$. Suppose Θ' is a truncation of Θ'' : $\Theta' \subseteq \Theta''$ and for any $j \in \Theta''$ with $\min \Theta' \leq j \leq \max \Theta'$, we have $j \in \Theta'$. Let $p_{\Theta''}$ and $p_{\Theta'}$ be distributions on Θ'' and Θ' respectively such that $p_{\Theta'}$ is the conditional distribution of $p_{\Theta''}$. If $p_{\Theta''}$ is logconcave, then $p_{\Theta'}$ is also logconcave.

PROOF. Since $p_{\Theta''}$ is logconcave, for all $i = \min \Theta' + 1, \dots, \max \Theta' - 1$,

$$\frac{p_{\Theta'}^i}{p_{\Theta'}^{i-1}} = \frac{\frac{p_{\Theta''}^i}{\sum_{j=\min \Theta'}^{\max \Theta'} p_{\Theta''}^j}}{\frac{p_{\Theta''}^{i-1}}{\sum_{j=\min \Theta'}^{\max \Theta'} p_{\Theta''}^j}} = \frac{p_{\Theta''}^i}{p_{\Theta''}^{i-1}} \geq \frac{p_{\Theta''}^{i+1}}{p_{\Theta''}^i} = \frac{\frac{p_{\Theta''}^{i+1}}{\sum_{j=\min \Theta'}^{\max \Theta'} p_{\Theta''}^j}}{\frac{p_{\Theta''}^i}{\sum_{j=\min \Theta'}^{\max \Theta'} p_{\Theta''}^j}} = \frac{p_{\Theta'}^{i+1}}{p_{\Theta'}^i},$$

where the first and last equality follows from $p_{\Theta'}$ being the conditional distribution of $p_{\Theta''}$. \square

Reverse Bayesian updating is essentially equality of relative likelihoods. More generally, we can consider ordering in relative likelihoods. The proof of the following lemma is similar to the proof of Lemma 8 but makes no use of logconcavity.

Lemma 11 Consider two distributions p and q on $\Theta = \{1, \dots, n\}$ such that p dominates q in relative likelihoods: For $i = 1, \dots, n - 1$,

$$\frac{p^{i+1}}{p^i} \geq (>) \frac{q^{i+1}}{q^i}.$$

Then:

(i) For all i, m with $i + m \leq n$,

$$\frac{p^{i+m}}{p^i} \geq (>) \frac{q^{i+m}}{q^i};$$

(ii) p dominates q in inverse hazard rates:

$$\frac{\sum_{m>i}^n p^m}{p^i} \geq (>) \frac{\sum_{m>i}^n q^m}{q^i} \text{ for } i = 1, \dots, n - 1.$$

PROOF. We have

$$\log p^{i+1} - \log p^i \geq (>) \log q^{i+1} - \log q^i \text{ for all } i = 1, \dots, n - 1.$$

(i) For $i, m = 1, \dots, n$, with $i + m \leq n$,

$$\begin{aligned} \frac{p^{i+m}}{p^i} &= \frac{p^{i+1}}{p^i} \frac{p^{i+2}}{p^{i+1}} \cdots \frac{p^{i+m}}{p^{i+m-1}} \\ \log \left(\frac{p^{i+m}}{p^i} \right) &= \log \left(\frac{p^{i+1}}{p^i} \frac{p^{i+2}}{p^{i+1}} \cdots \frac{p^{i+m}}{p^{i+m-1}} \right) \\ &= (\log p^{i+1} - \log p^i) + (\log p^{i+2} - \log p^{i+1}) + \dots + (\log p^{i+m} - \log p^{i+m-1}) \\ &\geq (>) (\log q^{i+1} - \log q^i) + (\log q^{i+2} - \log q^{i+1}) + \dots + (\log q^{i+m} - \log q^{i+m-1}) \end{aligned}$$

$$\begin{aligned}
&= \log \left(\frac{q^{i+1}}{q^i} \frac{q^{i+2}}{q^{i+1}} \cdots \frac{q^{i+m}}{q^{i+m-1}} \right) = \log \left(\frac{q^{i+m}}{q^i} \right) \\
&\frac{q^{i+1}}{q^i} \frac{q^{i+2}}{q^{i+1}} \cdots \frac{q^{i+m}}{q^{i+m-1}} = \frac{q^{i+m}}{q^i},
\end{aligned}$$

where the inequality follows from above inequality, applied to each term of the sum.

(ii) Rewrite (i) as

$$p^i q^{i+m} - q^i p^{i+m} \geq (>) 0.$$

Then

$$\begin{aligned}
&\sum_{m=1}^{n-i} (p^i q^{i+m} - q^i p^{i+m}) \geq (>) 0 \\
p^i \left(\sum_{m>i}^n q^m \right) - q^i \left(\sum_{m>i}^n p^m \right) &\geq (>) 0.
\end{aligned}$$

This completes the proof. \square

An implication of previous lemma and reverse Bayesianism is as follows:

Corollary 1 *Let $\Theta'' = \{1, \dots, n\}$. Suppose Θ' is a truncation of Θ'' : $\Theta' \subseteq \Theta''$ and for any $j \in \Theta''$ with $\min \Theta' \leq j \leq \max \Theta'$, we have $j \in \Theta'$. Let $p_{\Theta''}$ and $p_{\Theta'}$ be distributions on Θ'' and Θ' respectively such that $p_{\Theta'}$ is the conditional distribution of full-support distribution $p_{\Theta''}$. If $\max \Theta'' = \max \Theta'$, then for all $p_{\Theta''}$ and $p_{\Theta'}$ we have:*

(i) *For all $i \in \Theta'$ and m with $i + m \leq \max \Theta'$,*

$$\frac{p_{\Theta''}^{i+m}}{p_{\Theta''}^i} = \frac{p_{\Theta'}^{i+m}}{p_{\Theta'}^i};$$

(ii) *Inverse hazard rates are equal:*

$$\frac{\sum_{m>i}^{\max \Theta''} p_{\Theta''}^m}{p_{\Theta''}^i} = \frac{\sum_{m>i}^{\max \Theta'} p_{\Theta'}^m}{p_{\Theta'}^i} \text{ for } i = 1, \dots, \Theta' - 1.$$

Reverse Bayesianism allows us to compare the optimal menus across principal's awareness levels.

Lemma 12 *Let $\Theta'' = \{1, \dots, n\}$. Suppose Θ' is a truncation of Θ'' : $\Theta' \subseteq \Theta''$ and for any $j \in \Theta''$ with $\min \Theta' \leq j \leq \max \Theta'$, we have $j \in \Theta'$. Let $p_{\Theta''}$ and $p_{\Theta'}$ be distributions on Θ'' and Θ' respectively such that $p_{\Theta'}$ is the conditional distribution of logconcave full-support distribution $p_{\Theta''}$. For $i \in \Theta'$, let $\hat{q}_{\Theta''}^i$ and $\hat{q}_{\Theta'}^i$ denote the solutions for agent i of the principal's optimization problems w.r.t. $p_{\Theta''}$ and $p_{\Theta'}$, respectively.*

(i) *If $\max \Theta'' > \max \Theta'$, then for all $i \in \Theta'$, $\hat{q}_{\Theta''}^i < \hat{q}_{\Theta'}^i$.*

(ii) If $\max \Theta'' = \max \Theta'$, then for all $i \in \Theta'$, $\hat{q}_{\Theta''}^i = \hat{q}_{\Theta'}^i$.

PROOF. (i): Consider any $i \in \Theta'$ with $i < \max \Theta'$. Since by Lemma 10, $p_{\Theta'}$ is logconcave, and $v''(q) < 0$, we use first-order conditions to show:

$$\begin{aligned}
v'(\hat{q}_{\Theta''}^i) &> v'(\hat{q}_{\Theta'}^i) \\
\kappa^{(i)} + \frac{\sum_{j=i+1}^n p_{\Theta''}^j}{p_{\Theta''}^i} (\kappa^{(i)} - \kappa^{(i+1)}) &> \kappa^{(i)} + \frac{\sum_{j=i+1}^{\max \Theta'} p_{\Theta'}^{(j)}}{p_{\Theta'}^{(i)}} (\kappa^{(i)} - \kappa^{(i+1)}) \\
\frac{\sum_{j=i+1}^n p_{\Theta''}^{(j)}}{p_{\Theta''}^{(i)}} &> \frac{\sum_{j=i+1}^{\max \Theta'} p_{\Theta'}^{(j)}}{p_{\Theta'}^{(i)}} \\
\frac{\sum_{j=i+1}^{\max \Theta'} p_{\Theta''}^{(j)}}{p_{\Theta''}^{(i)}} + \frac{\sum_{j=\max \Theta'+1}^n p_{\Theta''}^{(j)}}{p_{\Theta''}^{(i)}} &> \frac{\sum_{j=i+1}^{\max \Theta'} p_{\Theta'}^{(j)}}{p_{\Theta'}^{(i)}} \\
\frac{\sum_{j=i+1}^{\max \Theta'} p_{\Theta'}^{(j)}}{p_{\Theta'}^{(i)}} + \frac{\sum_{j=\max \Theta'+1}^{|\Theta''|} p_{\Theta''}^{(j)}}{p_{\Theta''}^{(i)}} &> \frac{\sum_{j=i+1}^{\max \Theta'} p_{\Theta'}^{(j)}}{p_{\Theta'}^{(i)}} \\
\frac{\sum_{j=|\Theta''|+1}^{|\Theta''|} p_{\Theta''}^{(j)}}{p_{\Theta''}^{(i)}} &> 0,
\end{aligned}$$

where the second-to-last line follows from reverse Bayesianism.

Now consider $i = \max \Theta'$:

$$\begin{aligned}
v'(\hat{q}_{\Theta''}^{(\max \Theta')}) &> v'(\hat{q}_{\Theta'}^{(\max \Theta')}) \\
\kappa^{(\max \Theta')} + \frac{\sum_{j=\max \Theta'}^n p_{\Theta''}^{(j)}}{p_{\Theta''}^{(\max \Theta')}} (\kappa^{(\max \Theta')} - \kappa^{(\max \Theta'+1)}) &> \kappa^{(\max \Theta')} \\
\frac{\sum_{j=\max \Theta'}^n p_{\Theta''}^{(j)}}{p_{\Theta''}^{(\max \Theta')}} (\kappa^{(\max \Theta')} - \kappa^{(\max \Theta'+1)}) &> 0.
\end{aligned}$$

(ii) For $i = \min \Theta', \dots, n$,

$$\begin{aligned}
v'(\hat{q}_{\Theta''}^i) &= v'(\hat{q}_{\Theta'}^i) \\
\kappa^{(i)} + \frac{\sum_{j=i+1}^n p_{\Theta''}^j}{p_{\Theta''}^i} (\kappa^{(i)} - \kappa^{(i+1)}) &= \kappa^{(i)} + \frac{\sum_{j=i+1}^n p_{\Theta'}^{(j)}}{p_{\Theta'}^{(i)}} (\kappa^{(i)} - \kappa^{(i+1)}) \\
\frac{\sum_{j=i+1}^n p_{\Theta''}^j}{p_{\Theta''}^i} (\kappa^{(i)} - \kappa^{(i+1)}) &= \frac{\sum_{j=i+1}^n p_{\Theta'}^{(j)}}{p_{\Theta'}^{(i)}} (\kappa^{(i)} - \kappa^{(i+1)}),
\end{aligned}$$

where the last line follows now from reverse Bayesianism. \square

More generally, dominance of relative likelihoods allows us to compare solutions to the principal's optimization problems.

Lemma 13 Let $\Theta'' = \{1, \dots, n\}$. Suppose Θ' is a truncation of Θ'' : $\Theta' \subseteq \Theta''$ and for any $j \in \Theta''$ with $\min \Theta' \leq j \leq \max \Theta'$, we have $j \in \Theta'$. Let $p_{\Theta''}$ and $p_{\Theta'}$ be logconcave full-support distributions on Θ'' and Θ' . For $i \in \Theta'$, let $\hat{q}_{\Theta''}^i$ and $\hat{q}_{\Theta'}^i$ denote the solutions for agent i of the principal's optimization problems w.r.t. $p_{\Theta''}$ and $p_{\Theta'}$, respectively.

(i) If $p_{\Theta''}$ relative likelihood dominates $p_{\Theta'}$ on the latter's support and $\max \Theta'' \geq \max \Theta'$, then for all $i \in \Theta'$, $\hat{q}_{\Theta''}^i < \hat{q}_{\Theta'}^i$.

(ii) If $p_{\Theta'}$ relative likelihood dominates $p_{\Theta''}$ on the former's support and $\max \Theta'' = \max \Theta'$, then for all $i \in \Theta'$, $\hat{q}_{\Theta''}^i > \hat{q}_{\Theta'}^i$.

PROOF. (i): Consider first the case $\max \Theta'' > \max \Theta'$ and take $i \in \Theta'$ such that $i < \max \Theta'$. Since by Lemma 10, $p_{\Theta'}$ is logconcave and $v''(q) < 0$, we use first-order conditions to show:

$$\begin{aligned} v'(\hat{q}_{\Theta''}^i) &> v'(\hat{q}_{\Theta'}^i) \\ \kappa^{(i)} + \frac{\sum_{j=i+1}^n p_{\Theta''}^j}{p_{\Theta''}^i} (\kappa^{(i)} - \kappa^{(i+1)}) &> \kappa^{(i)} + \frac{\sum_{j=i+1}^{\max \Theta'} p_{\Theta'}^j}{p_{\Theta'}^i} (\kappa^{(i)} - \kappa^{(i+1)}) \\ \frac{\sum_{j=i+1}^n p_{\Theta''}^j}{p_{\Theta''}^i} &> \frac{\sum_{j=i+1}^{\max \Theta'} p_{\Theta'}^j}{p_{\Theta'}^i} \\ \frac{\sum_{j=i+1}^{\max \Theta'} p_{\Theta''}^j}{p_{\Theta''}^i} + \frac{\sum_{j=\max \Theta'+1}^n p_{\Theta''}^j}{p_{\Theta''}^i} &> \frac{\sum_{j=i+1}^{\max \Theta'} p_{\Theta'}^j}{p_{\Theta'}^i}. \end{aligned}$$

To see the last inequality, note that since $p_{\Theta''}$ relative likelihood dominates $p_{\Theta'}$ on latter's support,

$$\frac{\sum_{j=i+1}^{\max \Theta'} p_{\Theta''}^j}{p_{\Theta''}^i} > \frac{\sum_{j=i+1}^{\max \Theta'} p_{\Theta'}^j}{p_{\Theta'}^i}$$

follows from Lemma 11. For $i = \max \Theta'$, the same arguments as in the proof of Lemma 12 apply.

Now take the case $\max \Theta'' = \max \Theta'$. For $i = \min \Theta', \dots, n$, the term $\frac{\sum_{j=\max \Theta'+1}^n p_{\Theta''}^j}{p_{\Theta''}^i}$ does not exist in above inequality. This completes the proof of (i).

(ii) For $i = \min \Theta', \dots, n$,

$$\begin{aligned} v'(\hat{q}_{\Theta''}^i) &< v'(\hat{q}_{\Theta'}^i) \\ \kappa^{(i)} + \frac{\sum_{j=i+1}^n p_{\Theta''}^j}{p_{\Theta''}^i} (\kappa^{(i)} - \kappa^{(i+1)}) &< \kappa^{(i)} + \frac{\sum_{j=i+1}^n p_{\Theta'}^{(j)}}{p_{\Theta'}^{(i)}} (\kappa^{(i)} - \kappa^{(i+1)}) \\ \frac{\sum_{j=i+1}^n p_{\Theta''}^j}{p_{\Theta''}^i} &< \frac{\sum_{j=i+1}^n p_{\Theta'}^{(j)}}{p_{\Theta'}^{(i)}}, \end{aligned}$$

where the last line follows now from Lemma 11. \square

Remark 2 The proof of Lemma 12 reveals that $p_{\Theta''}$ relative likelihood dominating $p_{\Theta'}$ or the latter being the conditional of the former is not required for $\hat{q}_{\Theta''}^{\max \Theta'} < \hat{q}_{\Theta'}^{\max \Theta'}$ if $\max \Theta'' \geq \max \Theta'$.

The last lemma does not treat the case $\max \Theta'' > \max \Theta'$ when $p_{\Theta'}$ relative likelihood dominates in $p_{\Theta''}$ on the former's support. The reason is that the difference in upper bounds of the support makes it difficult to compare the distributions without further assumptions on inverse hazard rates.

B Proofs

Proof of Proposition 1

Let $\kappa^{(i)}$ denote the order statistics of marginal costs in space $\bar{\Theta}$, where $\kappa^{(1)}$ refers to the highest marginal cost.

Level 1. *Principal:* At her information set h_P , any first-level Δ -prudent rationalizable strategy s_P of the principal assigns a menu of contracts

$$\left\{ \left(q_{\Theta_{h_P}}^{(i)}, t_{\Theta_{h_P}}^{(i)} \right) \right\}_{i=1, \dots, |\Theta_{h_P}|} \in ([0, b]^2)^{|\Theta_{h_P}|}$$

that maximize expected profits w.r.t. some full support belief on types $\kappa^{(1)}, \dots, \kappa^{(|\Theta_{h_P}|)}$. (Since the principal's beliefs are full support over strategies and marginal cost types, her marginal on marginal cost types must be full support at this level.)

Agent: Consider any last information set of the agent in tree T_{Θ} . Note that these information sets are singleton. Thus, the agent is certain of the menu offered by the principal. If s_A is a first-level rationalizable strategy of the agent, it selects at this information set a contract $(q, t) \in s_P(h_P)$ that maximizes expected payoff given the incentive and participation constraints. If none of the contracts in $s_P(h_P)$ satisfies the participation constraint, he selects the outside option. Since the agent's payoff function satisfies decreasing differences (Lemma 2), the selected contract quantity is decreasing in the marginal cost type of the agent (Lemma 3).

At any first information sets of the agent in tree T_{Θ} , any strategy is first-level prudent rationalizable for any marginal cost type $\kappa^{(i)}$ with $i = 1, \dots, |\Theta|$. Disclosure is rational if he believes with sufficiently large probability that the principal will offer a better deal after disclosure. Non-disclosure is optimal if he believes with sufficiently high probability that disclosure will not lead to a better deal.

Level 2. *Principal:* The principal is certain of first-level Δ -prudent rationalizable strategies of the agent. Thus, she is certain that the agent observes participation constraints and self-selects among contracts according to his incentives. Moreover, she is certain that the agent's chosen quantities are monotone in the agent's marginal cost type.

First-level Δ -prudent rationalizability imposes no restrictions on the agent's strategies w.r.t. to raising the principal's awareness. Since her belief system is cautious (i.e., full support beliefs), she believes at any of her information sets at which she became aware that any type that she is now aware of could have raised her awareness. If Θ is the set of types that the principal is aware of at the information set, denote by p_{Θ}^i the principal's marginal probability that the agent's type has marginal cost $\kappa^{(i)}$ with $i = 1, \dots, |\Theta|$. Since p_{Θ} logconcave and full support,

the principal's optimization problem and its solution is given as in Appendix A for any Θ . Moreover, if $\Theta' \subseteq \Theta''$, then $p_{\Theta'}$ and $p_{\Theta''}$ satisfy reverse Bayesianism since the principal believes at any of her information sets that any type could have raised her awareness. Thus, quantities for each type are monotone across trees in which he exists (Lemma 12 (i)).

Agent: No additional strategies are eliminated at level 2.

Level 3. Principal: No additional strategies are eliminated at level 3 since no other strategies of the agents have been eliminated at level 2.

Agent: The agent is certain of level-2 Δ -prudent rationalizable strategies of the principal. Type $\kappa^{(1)}$ is indifferent between disclosing any Θ and not disclosing, since he is always held to his outside option with any level-2 Δ -rationalizable strategy of the principal. Thus, by the tie-breaking assumption, he does not disclose. We show by induction on order statistics of types that all $i = 2, \dots, |\Theta'|$ prefer not to disclose $\Theta'' \not\subseteq \Theta'$.

Base case $i = 2$:

$$\begin{aligned} u_A(\hat{q}_{\Theta''}^2, \hat{t}_{\Theta''}^2, \kappa^{(2)}) &< u_A(\hat{q}_{\Theta'}^2, \hat{t}_{\Theta'}^2, \kappa^{(2)}) \\ \hat{t}_{\Theta''}^2 - \kappa^{(2)} \hat{q}_{\Theta''}^2 &< \hat{t}_{\Theta'}^2 - \kappa^{(2)} \hat{q}_{\Theta'}^2 \\ \hat{t}_{\Theta''}^1 - \kappa^{(2)} \hat{q}_{\Theta''}^1 &< \hat{t}_{\Theta'}^1 - \kappa^{(2)} \hat{q}_{\Theta'}^1 \\ \kappa^{(1)} \hat{q}_{\Theta''}^1 - \kappa^{(2)} \hat{q}_{\Theta''}^1 &< \kappa^{(1)} \hat{q}_{\Theta'}^1 - \kappa^{(2)} \hat{q}_{\Theta'}^1 \\ (\kappa^{(1)} - \kappa^{(2)}) \hat{q}_{\Theta''}^1 &< (\kappa^{(1)} - \kappa^{(2)}) \hat{q}_{\Theta'}^1 \end{aligned}$$

follows from $\hat{q}_{\Theta''}^1 < \hat{q}_{\Theta'}^1$ (Lemma 12 (i)). (The third line follows from the incentive compatibility constraints. The fourth line follows from the participation constraint of marginal cost type $\kappa^{(1)}$.)

Induction hypothesis:

$$u_A(\hat{q}_{\Theta''}^i, \hat{t}_{\Theta''}^i, \kappa^{(i)}) < u_A(\hat{q}_{\Theta'}^i, \hat{t}_{\Theta'}^i, \kappa^{(i)}).$$

Inductive step: For i with $1 < i < |\Theta'|$,

$$\begin{aligned} u_A(\hat{q}_{\Theta''}^{i+1}, \hat{t}_{\Theta''}^{i+1}, \kappa^{(i+1)}) &< u_A(\hat{q}_{\Theta'}^{i+1}, \hat{t}_{\Theta'}^{i+1}, \kappa^{(i+1)}) \\ \hat{t}_{\Theta''}^{i+1} - \kappa^{(i+1)} \hat{q}_{\Theta''}^{i+1} &< \hat{t}_{\Theta'}^{i+1} - \kappa^{(i+1)} \hat{q}_{\Theta'}^{i+1} \\ \hat{t}_{\Theta''}^i - \kappa^{(i+1)} \hat{q}_{\Theta''}^i &< \hat{t}_{\Theta'}^i - \kappa^{(i+1)} \hat{q}_{\Theta'}^i \\ \hat{t}_{\Theta''}^i - \kappa^{(i)} \hat{q}_{\Theta''}^i + \kappa^{(i)} \hat{q}_{\Theta''}^i - \kappa^{(i+1)} \hat{q}_{\Theta''}^i &< \hat{t}_{\Theta'}^i - \kappa^{(i)} \hat{q}_{\Theta'}^i + \kappa^{(i)} \hat{q}_{\Theta'}^i - \kappa^{(i+1)} \hat{q}_{\Theta'}^i \\ u_A(\hat{q}_{\Theta''}^i, \hat{t}_{\Theta''}^i, \kappa^{(i)}) + (\kappa^{(i)} - \kappa^{(i+1)}) \hat{q}_{\Theta''}^i &< u_A(\hat{q}_{\Theta'}^i, \hat{t}_{\Theta'}^i, \kappa^{(i)}) + (\kappa^{(i)} - \kappa^{(i+1)}) \hat{q}_{\Theta'}^i \end{aligned}$$

follows now from the induction hypothesis and $\hat{q}_{\Theta''}^i < \hat{q}_{\Theta'}^i$ (Lemma 12 (i)).

Next, we show that marginal cost type $|\Theta'| + 1$ prefers not to disclose:

$$\begin{aligned} u_A(\hat{q}_{\Theta''}^{|\Theta'|+1}, \hat{t}_{\Theta''}^{|\Theta'|+1}, \kappa^{(|\Theta'|+1)}) &< u_A(\hat{q}_{\Theta'}^{|\Theta'|}, \hat{t}_{\Theta'}^{|\Theta'|}, \kappa^{(|\Theta'|+1)}) \\ \hat{t}_{\Theta''}^{|\Theta'|+1} - \kappa^{(|\Theta'|+1)} \hat{q}_{\Theta''}^{|\Theta'|+1} &< \hat{t}_{\Theta'}^{|\Theta'|} - \kappa^{(|\Theta'|+1)} \hat{q}_{\Theta'}^{|\Theta'|} \\ \hat{t}_{\Theta''}^{|\Theta'|} - \kappa^{(|\Theta'|+1)} \hat{q}_{\Theta''}^{|\Theta'|} &< \hat{t}_{\Theta'}^{|\Theta'|} - \kappa^{(|\Theta'|+1)} \hat{q}_{\Theta'}^{|\Theta'|} \\ \hat{t}_{\Theta''}^{|\Theta'|} - \kappa^{(|\Theta'|)} \hat{q}_{\Theta''}^{|\Theta'|} + \kappa^{(|\Theta'|)} \hat{q}_{\Theta''}^{|\Theta'|} - \kappa^{(|\Theta'|+1)} \hat{q}_{\Theta''}^{|\Theta'|} &< \end{aligned}$$

$$\begin{aligned}
& \hat{t}_{\Theta'}^{|\Theta'|} - \kappa^{(|\Theta'|)} \hat{q}_{\Theta'}^{|\Theta'|} + \kappa^{(|\Theta'|)} \hat{q}_{\Theta'}^i - \kappa^{(|\Theta'|+1)} \hat{q}_{\Theta'}^{|\Theta'|} \\
u_A(\hat{q}_{\Theta''}^{|\Theta'|}, \hat{t}_{\Theta''}^{|\Theta'|}, \kappa^{(|\Theta'|)}) + (\kappa^{(|\Theta'|)} - \kappa^{(|\Theta'|+1)}) \hat{q}_{\Theta''}^{|\Theta'|} & < \\
& u_A(\hat{q}_{\Theta'}^{|\Theta'|}, \hat{t}_{\Theta'}^{|\Theta'|}, \kappa^{(|\Theta'|)}) + (\kappa^{(|\Theta'|)} - \kappa^{(|\Theta'|+1)}) \hat{q}_{\Theta'}^{|\Theta'|},
\end{aligned}$$

which follows from the previous inductive proof and $\hat{q}_{\Theta''}^{|\Theta'|} < \hat{q}_{\Theta'}^{|\Theta'|}$ (Lemma 12 (i)). It now follows immediately that any i with $|\Theta'| + 1 \leq i < |\Theta''|$ prefers to raise the principal's awareness at most to $\Theta_{(i-1)}$, where $\Theta_{(i-1)}$ denotes the space of moves of nature in which $i - 1$ is the lowest marginal cost type.

Level 4. Principal: If the principal is not made aware of additional marginal cost types (i.e., her information set in the lowest tree), then no further restrictions are imposed by fourth-level Δ -prudent rationalizability. That is, she offers a menu of contracts that maximize expected utility w.r.t. a full support belief over marginal cost types she has been aware of subject to the participation constraint for marginal cost type $\kappa^{(1)}$ and incentive compatibility constraints for all others.

If the principal's awareness is raised to $\Theta_{(i)}$, then since the principal is now certain of level-3 Δ -prudent rationalizable strategies of the agent, she realizes that no type in $\Theta_{(i)}$ has had an incentive to make her aware of $\Theta_{(i)}$. (Any type in $i \in \Theta_{(i)} \setminus \Theta_P$ would have raised her awareness at most only to $\Theta_{(i-1)}$.) Thus, she cannot further rationalize the agent's action. Hence, she is allowed to believe anything according to some belief system in B_P and her set of strategies is not refined further: $R_P^4 = R_P^3$.

Agents: No additional strategies can be eliminated at level 4 since there were none eliminated for the principal at level 3.

Since none of the players' sets of strategies are refined at level 4, none are refined at further levels. This completes the proof. \square

Proof of Proposition 2

Level 1. This part of the proof is analogous to that of Proposition 1.

Level 2. This part of the proof is also analogous to that of Proposition 1. Compared to the proof of Proposition 1, however, it is convenient to use the dual order on the marginal cost types. That is, we now denote by $\kappa^{(1)}$ the type with the lowest marginal costs and by $\kappa^{(|\Theta_P|)}$ the type with the highest marginal costs of which the principal is initially aware. This makes the order statistics of marginal cost types in lower spaces invariant to raising awareness of additional types with higher marginal costs. The first-order conditions can be obtained from Appendix A with the appropriate adjustments in notation for the dual order statistics.

Unlike in the proof of Proposition 1, reverse Bayesianism implies now (Lemma 12 (ii)) that for $i = 1, \dots, |\Theta'|$ and any $\Theta'' \supseteq \Theta'$

$$\hat{q}_{\Theta''}^i = \hat{q}_{\Theta'}^i.$$

Level 3. Principal: No additional strategies are eliminated at level 3 since no further strategies of the agents have been eliminated at level 2.

Agent: Any level-3 Δ prudent rationalizable strategy of the agent must satisfy the following:

(a) Type $\kappa^{(|\bar{\Theta}|)}$ is indifferent between disclosing or not disclosing $\bar{\Theta}$ to the principal since he is always held to his outside option. When not disclosing, his participation constraint is violated and he selects the outside option. When disclosing, he receives the same payoff as from taking the outside option.

(b) We show by induction on the dual order statistics of agent types that all $i = 1, \dots, |\bar{\Theta}| - 1$ prefer to disclose $\bar{\Theta}$ over any other Θ .

Base case $i = |\bar{\Theta}| - 1$: For any $\Theta_j = \{1, \dots, j\}$ (i.e., all marginal cost types from the lowest marginal cost type to marginal cost type $\kappa^{(j)}$) with $j \leq |\bar{\Theta}| - 1$,

$$\begin{aligned} u_A(\hat{q}_{\bar{\Theta}}^{|\bar{\Theta}|-1}, \hat{t}_{\bar{\Theta}}^{|\bar{\Theta}|-1}, \kappa^{(|\bar{\Theta}|-1)}) &\geq u_A(\text{raise awareness to } \Theta_j, \kappa^{(|\bar{\Theta}|-1)}) \\ \hat{t}_{\bar{\Theta}}^{|\bar{\Theta}|-1} - \kappa^{(|\bar{\Theta}|-1)} \hat{q}_{\bar{\Theta}}^{|\bar{\Theta}|-1} &\geq 0 \\ \hat{t}_{\bar{\Theta}}^{|\bar{\Theta}|} - \kappa^{(|\bar{\Theta}|-1)} \hat{q}_{\bar{\Theta}}^{|\bar{\Theta}|} &\geq 0, \end{aligned}$$

where the r.h.s. of the second line follows from the fact that type $\kappa^{(|\bar{\Theta}|-1)}$ would select the outside option when the principal is aware only of Θ_j if $j < |\bar{\Theta}| - 1$ or be held to the payoff of the outside option if $j = |\bar{\Theta}| - 1$. The last line follows from incentive compatibility as well as the highest-marginal cost type's participation constraint.

Induction hypothesis: For any i with $1 < i < |\bar{\Theta}|$,

$$u_A(\hat{q}_{\bar{\Theta}}^{i+1}, \hat{t}_{\bar{\Theta}}^{i+1}, \kappa^{(i+1)}) \geq u_A(\hat{q}_{\Theta_j}^{i+1}, \hat{t}_{\Theta_j}^{i+1}, \kappa^{(i+1)}).$$

Inductive step: We prove for i with $1 \leq i < |\bar{\Theta}|$ that:

$$u_A(\hat{q}_{\bar{\Theta}}^i, \hat{t}_{\bar{\Theta}}^i, \kappa^{(i)}) \geq u_A(\hat{q}_{\Theta_j}^i, \hat{t}_{\Theta_j}^i, \kappa^{(i)}).$$

If $j < i$, then disclosing just Θ_j results in i taking the outside option and there is nothing to prove. If $j = i$, then disclosing just Θ_j results in i being held to his outside option and the payoff is zero. Again, there is nothing to prove. Next, consider the case $j > i$:

$$\begin{aligned} u_A(\hat{q}_{\bar{\Theta}}^i, \hat{t}_{\bar{\Theta}}^i, \kappa^{(i)}) &\geq u_A(\hat{q}_{\Theta_j}^i, \hat{t}_{\Theta_j}^i, \kappa^{(i)}) \\ \hat{t}_{\bar{\Theta}}^i - \kappa^{(i)} \hat{q}_{\bar{\Theta}}^i &\geq \hat{t}_{\Theta_j}^i - \kappa^{(i)} \hat{q}_{\Theta_j}^i \\ \hat{t}_{\bar{\Theta}}^{i+1} - \kappa^{(i)} \hat{q}_{\bar{\Theta}}^{i+1} &\geq \hat{t}_{\Theta_j}^{i+1} - \kappa^{(i)} \hat{q}_{\Theta_j}^{i+1} \\ \hat{t}_{\bar{\Theta}}^{i+1} - \kappa^{(i+1)} \hat{q}_{\bar{\Theta}}^{i+1} + \kappa^{(i+1)} \hat{q}_{\bar{\Theta}}^{i+1} - \kappa^{(i)} \hat{q}_{\bar{\Theta}}^{i+1} &\geq \hat{t}_{\Theta_j}^{i+1} - \kappa^{(i+1)} \hat{q}_{\Theta_j}^{i+1} + \kappa^{(i+1)} \hat{q}_{\Theta_j}^{i+1} - \kappa^{(i)} \hat{q}_{\Theta_j}^{i+1} \\ u_A(\hat{q}_{\bar{\Theta}}^{i+1}, \hat{t}_{\bar{\Theta}}^{i+1}, \kappa^{(i+1)}) + (\kappa^{(i+1)} - \kappa^{(i)}) \hat{q}_{\bar{\Theta}}^{i+1} &\geq u_A(\hat{q}_{\Theta_j}^{i+1}, \hat{t}_{\Theta_j}^{i+1}, \kappa^{(i+1)}) + (\kappa^{(i+1)} - \kappa^{(i)}) \hat{q}_{\Theta_j}^{i+1} \end{aligned}$$

which follows now from the induction hypothesis, $\kappa^{(i+1)} > \kappa^{(i)}$, and from the fact that $\hat{q}_{\bar{\Theta}}^{i+1} = \hat{q}_{\Theta_j}^{i+1}$ (Lemma 12 (ii)).

Level 4. Principal: At level 4, the principal is now certain of level-3 Δ -prudent rationalizable strategies of the agent. Thus, upon becoming aware of more types, she is certain that all agents could have raised her awareness. (Full support beliefs become now crucial for not ruling out

that the high cost type could have made her aware.) She cannot exclude any type and there is no further reduction of her strategy set.

Agent: Since the principal's set of strategies was not reduced at level 3, no strategies of the agent are eliminated at level 4.

Since none of the player's sets of strategies were refined at level 4, none are refined at further levels. This completes the proof. \square

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