A Note on Stochastic Orders and Incentives

Alejandro Francetich[*]

UW Bothell School of Business

January 29, 2026

Abstract

In contract design, a profit-maximizing principal trades off social surplus for lower information rents. Imagine that the principal is able to influence the distribution of agent types; for instance, a monopolist can invest in marketing campaigns to boost demand. Changes in the type distribution that generate more social surplus, however, may not be profitable for the principal if they lead to even higher information rents. When are the social and private benefits aligned?

In a quasilinear setting, Proposition 2 in Hart and Reny (2015) can be adapted to show that first-order stochastic dominance (FOSD) guarantees said alignment by giving monotonicity of the principal's expected profit. This argument does not invoke the structure of expected information rents, nor whether incentive compatibility binds. With linear utilities, we propose a weaker stochastic order corresponding to said monotonicity: *incentive dominance* (ID), dominance in the increasing convex order (ICxOD) applied to possibly-truncated (excluding lower types), possibly-ironed virtual utilities. It turns out that, while weaker, ID is "very close" to FOSD: We show that, for regular distributions, FOSD is in fact equivalent to ICxOD applied to the non-truncated virtual utilities.

Keywords: Mechanism design; screening; information rents; stochastic dominance

JEL Classification Numbers: D82, D86

1 Introduction

A monopolist can invest in a marketing campaign to boost demand. If the buyer's reservation price is private information, a demand boost can take the form of a shift in the distribution of reservation prices: Ads may lead to shifting probability mass to higher values, or expanding the support upwards. In such contract-design problems, a payoff-maximizing principal trades off social surplus for lower information rents. Changes in the distribution of agent types may increase the social surplus generated, but they could also raise the amount due the agent in information rents. *When are social and private benefits aligned, in the sense that an improvement in the type distribution is also profitable for the principal when accounting for the impact of information rents?*

Hart and Reny (2015) shows that expected revenue under any mechanism where the agent's ex-post payments are monotonic with respect to first-order stochastic dominance (FOSD); in particular, this applies to the revenue-maximizing mechanism (Proposition 2). In a quasilinear setting, their argument can be adapted to expected profit maximization with strictly-convex costs. Thus, we have an answer to our question: If the shift in the type distribution constitutes an improvement in the sense of FOSD, such a social improvement is also privately profitable.

This argument does not invoke the structure of expected information rents in any way, nor even whether any of the incentive-compatibility constraints ever bind—that is, whether any of the distributions need to be ironed. This leaves room the follow-up question: If we further exploit the structure of the principal's problem, can we identify a "tighter" stochastic order that yields the desired motonicity result?

We answer the latter question in the context of linear utilities. For the complete information benchmark, increases in maximum expected social surplus correspond to dominance in the increasing convex order (ICxOD; Shaked and Shanthikumar 2007). Under asymmetric information, we introduce *incentive dominance* (ID), which is ICxOD applied to possibly-truncated (i.e., excluding lower types), possibly-ironed virtual utilities. This dominance criterion corresponds to monotonicity of maximum expected profit.

As expected, we show that FOSD is stronger that ID. While the reciprocal is not true, it turns out ID is nonetheless "very close" to FOSD: We prove that, for regular distributions, FOSD is equivalent to ICxOD applied to the non-truncated virtual utilities.

The rest of the paper is organized as follows. Section 2 presents an example that embodies the main incentive tension of the problem. Section 3 describes the

quasi-linear setting and shows that FOSD is a sufficient condition for monotonicity of profit. Section 4 further restricts attention to the linear setting and presents the incentive order, providing the connection between this concept and FOSD. Finally, section 5 concludes. A known, supplementary result is included in the appendix. The MATLAB code for the examples is available from the author upon request.

2 Example: Social vs. Private Benefit of Demand Boosts

A monopolist (she) produces and sells a good to a buyer (he). Production costs are given by $c(q) = \frac{q^2}{2}$. The buyer's utility is linear and his valuation v for the good is drawn uniformly from $[0, 1]$; his outside option is 0.

In the symmetric-information benchmark and by setting transfers equal to $t(v) = v \cdot q(v)$, the monopolist captures all of the social surplus, $W = \int_0^1 [v \cdot q(v) - c(q(v))] dv$, for herself as profit. Thus, the optimal mechanism is the first-best $q^{FB}(v) = v$ and the surplus generated is $W^* = \int_0^1 \frac{v^2}{2} dv = \frac{1}{6}$. With the buyer's valuation being private, the seller must provide incentives in the form of information rents. Accounting for the latter, the seller's expected profit is:

$$\Pi = \int_0^1 [(2v - 1) \cdot q(v) - c(q(v))] dv.$$

The optimal implementable mechanism is $q^*(v) = \max\{2v - 1, 0\}$ and the seller's maximum expected profit is $\Pi^* = \frac{1}{12}$.

Imagine now that the seller can effect a demand boost that results in a shift of the valuation distribution to a quadratic cdf, $F(v) = v^2$, thereby shifting probability mass uniformly to higher values. Ignoring the cost of enacting this shift and since the first-best allocation remains the same, the new maximum expected social surplus is $W^* = \int_0^1 \frac{v^2}{2} \cdot 2v dv = \frac{1}{4} > \frac{1}{6}$. Under private information, the new optimal mechanism and maximum profit are:

$$q^*(v) = \max \left\{ \frac{3v}{2} - \frac{1}{2v}, 0 \right\}, \quad \Pi^* = \int_{\frac{1}{\sqrt{3}}}^1 \frac{1}{2} \left(\frac{3\theta}{2} - \frac{1}{2\theta} \right)^2 \cdot 2\theta d\theta = \frac{\ln(3)}{8} > \frac{1}{12},$$

respectively. Thus, the shift is both socially benefitial and profitable for the seller.

Consider the following alternative scenario. The initial distribution for v is Beta$(5, 5)$

and the shifted distribution is Beta$(4, 4)$. The expected maximum surplus is:

$$W^* = \int_0^1 \frac{v^2}{2} \cdot 630 v^4 (1-v)^4 \mathrm{d}v = \frac{3}{22} \approx 0.1364$$

before the shift, and:

$$W^* = \int_0^1 \frac{v^2}{2} \cdot 140 v^3 (1-v)^3 \mathrm{d}v = \frac{5}{36} \approx 0.1389$$

afterwards. Thus, the shift is socially benefitial. This shift moves probability mass to the extremes of the support but in a way that raises social surplus.

Under asymmetric information, the allocation rules under the two distributions are presented in Figure 1. The maximum expected profits are $\Pi^* \approx 0.07139$ before and $\Pi^* \approx 0.07108$ after. Even if shifting the distribution were costless, this socially-beneficial shift leaves the seller worse off. Intuitively, the information rents due the buyer increase by more than the social surplus generated.
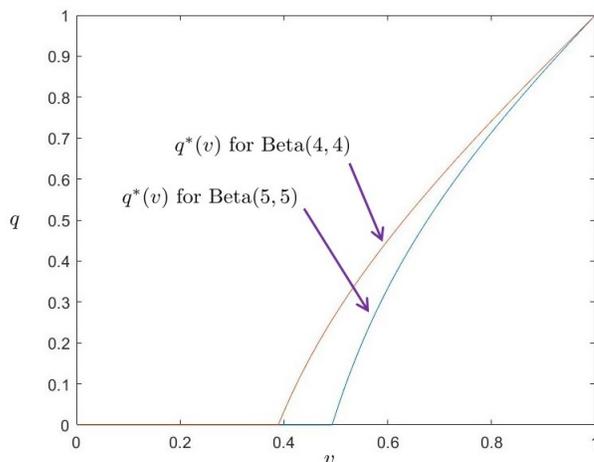

Figure 1: Allocation rules $q^*(v)$ for $v \sim \text{Beta}(5, 5)$ (blue) and $v \sim \text{Beta}(4, 4)$ (red).

3   Quasilinear Utility and FOSD

In the first version of the example from the previous section, the shift in the distribution (from linear to quadratic cdf) represents a shift in the sense of FOSD. In this section, in the setting of quasilinear utility, we adapt the argument from Proposition 2 in Hart and Reny (2015) to show that maximum expected profit is

non-decreasing with respect to FOSD—as is maximum expected social surplus.

A monopolist (the principal, she) produces and sells $q \in Q$ units of her product to a consumer (the agent, he) in exchange for payment $t \in T$. We assume that $Q = [0, K]$ and $T = [-L, L]$ for some large $K, L > 0$. The cost for the principal to produce q units is given by the cost function $c(q)$, so her profit is $\pi(t, q) = t - c(q)$. The agent's gross utility for the good is given by the function $v(q, v)$, where $v \geq 0$ is the agent's type. (The type space will be specified below, but we take it to be a subset of $\mathbb{R}_+$ for consistency with the linear setting in section 4.) His payoff is thus $u(q, t, v) = v(q, v) - t$.

We assume that $c(q)$ is strictly increasing, convex, and twice differentiable; and we also impose the usual conditions on $v(q, t)$:

1. For every $v \geq 0$, $v(0, v) = 0$ and $v(\cdot, v)$ is strictly increasing and concave;

2. For every $q \in Q$, $v(q, \cdot)$ is non-decreasing and continuously differentiable;

3. There exists an $M > 0$ such that $|v'_v(q, \cdot)| \leq M$ for all $q \in Q$.

4. *Single crossing property* (SCP): (a) For every $v \geq 0$, $v(\cdot, v)$ is differentiable, and: (b) For all $q \in Q$, $v'_q(q, \cdot)$ is strictly increasing.

There is a parametric family of probability distributions for the agent's valuation, $\mathcal{F} := \{F_\theta(v) : \theta \in \Theta\}$ for some parameter space Θ, with $F_\theta(v)$ denoting the cdf of the distribution identified by $\theta \in \Theta$. We assume that, for each $\theta \in \Theta$, F_θ is absolutely continuous and has full support on some interval $V_\theta := [\underline{v}_\theta, \overline{v}_\theta]$, where $0 \leq \underline{v}_\theta < \overline{v}_\theta$, with pdf $f_\theta(v)$. Finally, let $\succsim$ denote an order on $\mathcal{F}$. Slightly abusing notation, we identify $\succsim$ with its projection on both Θ and on the space of random variables, so write $\theta' \succsim \theta$ or $v|_{\theta'} \succsim v|_\theta$ for $F_{\theta'} \succsim F_\theta$.

A mechanism consists of an allocation function $q(v)$ and a transfer function $t(v)$. Although the relevant implementability constraints given $\theta \in \Theta$ apply only to types in the support V_θ, it will be helpful to be able to talk about mechanisms separately from the said distribution; i.e., to "decouple" the distribution's support and the mechanism's domain. Thus, we assume that there is a type set $V \subseteq \mathbb{R}_+$ such that $V_\theta \subseteq V$ for all $\theta \in \Theta$, and we take V to be the domain of the mechanisms in the notation.

Given $\theta \in \Theta$, the principal's problem is to design an implementable mechanism to maximize their expected profit:

$$\Pi(q, t, \theta) := \int_{V_\theta} \pi(v) f_\theta(v) \mathrm{d}v,$$

where, abusing notation, $\pi(v) := \pi(t(v), q(v))$.

Define the social surplus function $w : Q \times \mathbb{R}_+ \to \mathbb{R}$ as $w(q, v) := v(q, v) - c(q)$, and the maximum social surplus function $w^* : \mathbb{R}_+ \to \mathbb{R}$ as $w^*(v) := \sup(\{w(q, v) : q \in Q\})$. Denote the first-best allocation rule by $q^{FB}(v)$, so that $w^*(v) = w(q^{FB}(v), v)$. Let $W^*(\theta)$ be the expected maximum surplus under θ: $W^*(\theta) = \int_{V_\theta} w^*(v) f_\theta(v) \mathrm{d}v$.[1] In the full-information benchmark, by setting transfers to $t^{FB}(v) = v(q^{FB}(v), v)$, the principal captures the entire social surplus:

$$\Pi(q, t, \theta) \le W^*(\theta) = \Pi\left(q^{FB}, t^{FB}, \theta\right).$$

Under asymmetric information, she must provide the agent the right incentives. Given a type distribution $\theta \in \Theta$ and denoting the agent's truthful payoff by $U(v) := u(q(v), t(v), v)$, mechanism (q, t) is implementable if and only if:[2]

- $q(v)$ is non-decreasing on V_θ;

- For every $v \in V_\theta$, $U(v) = U(\underline{v}_\theta) + \int_{\underline{v}_\theta}^v v'_v(q(x), x) \, \mathrm{d}x$;

- $U(\underline{v}_\theta) \ge 0$, where 0 is the agent's outside option.

In what follows, in order to simplify the formulas and since our ultimate goal is the profit-maximizing mechanism, we set $U(\underline{v}_\theta) = 0$. Our space of implementable mechanisms given θ, $\mathcal{M}_\theta$, is thus:

$$\mathcal{M}_\theta := \left\{ (q, t) : q|_{V_\theta}(v) \text{ is non-decreasing and, } \forall v \in V_\theta, U(v) = \int_{\underline{v}_\theta}^v v'_v(q(x), x) \, \mathrm{d}x \right\}$$

We are interested in the behavior of the expected maximum-profit function, $\Pi^* : \Theta \to \mathbb{R}$, given by:

$$\Pi^*(\theta) := \sup(\{\Pi(q, t, \theta) : (q, t) \in \mathcal{M}_\theta\}).$$

Example 1. As before, take $c(q) = \frac{q^2}{2}$ and consider the case of linear utilities: $v(q, v) = v \cdot q$. Let $\Theta = [1, 2]$ and take $v \sim U[0, \theta]$ for every $\theta \in \Theta$. The optimal mechanism is given by $q^*_\theta(v) = \max\{2v - \theta, 0\}$ and maximum expected profit is $\Pi^*(\theta) = \frac{\theta^2}{12}$. $\triangle$

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[1] Notice that $q^{FB}(v)$ is independent of θ, except for its support V_θ.

[2] See Dewatripont and Bolton (2005); Segal (2010); Krishna (2010); Laffont and Martimort (2001), among many others.

Example 2. Take now $\Theta = (-1, 1]$ and, for each $\theta \in \Theta$, let $V_\theta = [0, 1]$ and $F_\theta(v) = v^{1+\theta}$. The optimal mechanism is now given by:

$$q_\theta^*(v) = \max\left\{\frac{(2+\theta)v^{1+\theta} - 1}{(1+\theta)v^\theta}, 0\right\};$$

Defining $\lambda_\theta := (2+\theta)^{-\frac{1}{1+\theta}}$, maximum expected profit is:

$$\Pi^*(\theta) = \frac{1+\theta}{2(1-\theta)(\theta+3)}\left[1 - \left(\frac{1}{2+\theta}\right)^{\frac{1-\theta}{1+\theta}}\right].$$

Figure 2 depicts the curve of $\Pi^*(\theta)$. △

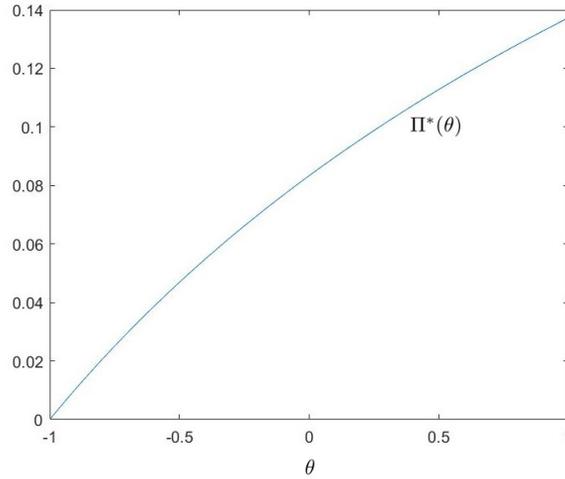Figure 2: Curve for $\Pi^*(\theta)$ in Example 2.

Example 3. Finally, consider $\Theta = (1, +\infty)$ and, for each $\theta \in \Theta$, let $V_\theta = [0, 1]$ and $F_\theta(v) = 1 - (1-v)^{\frac{1}{\theta-1}}$. Here, $q_\theta^*(v) = \max\{\theta v - \theta + 1, 0\}$ and:

$$\Pi^*(\theta) = (\theta - 1)\theta^{-\frac{\theta}{\theta-1}}.$$

Figure 3 shows the $\Pi^*(\theta)$ curve. △

In Examples 1 through 3, we have that $\theta' > \theta$ implies $\theta' \succsim_{FOSD} \theta$. To establish that the latter is a sufficient condition for monotonicity of $\Pi^*(\theta)$, we adapt Proposition 2
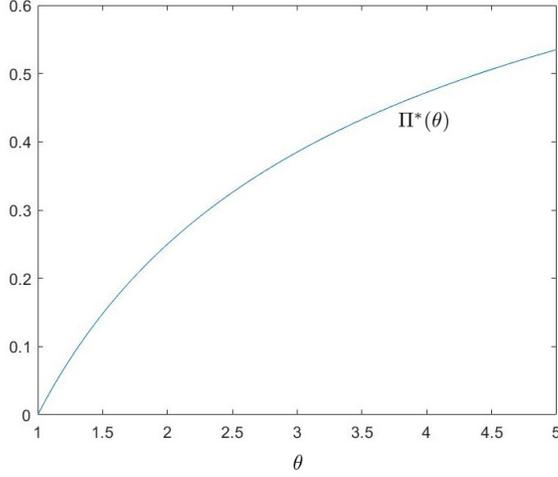
Figure 3: Curve for $\Pi^*(\theta)$ in Example 3.

in Hart and Reny (2015).[3]

Lemma 1 (Monotonicity of ex-post profit)**.** *Let (q, t) be an implementable mechanism mechanism such that $q(v) \leq q^{FB}(v)$ for every $v \in V$. Then, $\pi(v)$ is non-decreasing.*

Proof. By implementability, $\pi(v) = \nu(q(v), v) - c(q(v)) - \int_{\underline{v}}^{v} \nu_v'(q(x), x) \mathrm{d}x$. In this setting, incentive compatibility implies that $U(v)$ is differentiable almost everywhere, and thus so are $q(v)$ and $\pi(v)$. At any differentiability point, we have $\pi'(v) = [\nu_q'(q(v), v) - c'(q(v))] \cdot q'(v)$. The assumption that $q(v) \leq q^{FB}(v)$, along with the monotonicity condition for $q(v)$, yields $\pi'(v) \geq 0$. Thus, for any $v, v' \geq 0$ such that $v' > v$, $\pi(v') - \pi(v) = \int_{v}^{v'} \pi'(x) \mathrm{d}x \geq 0$. □

Proposition 1 (FOSD monotonicity)**.** *For every $\theta, \theta' \in \Theta$ such that $\theta' \succsim_{FOSD} \theta$,*

1. *$W^*(\theta') \geq W^*(\theta)$;*

2. *For every incentive-compatible mechanism (q, t) such that $q(v) \leq q^{FB}(v)$ for every $v \in V$, and for every $\theta, \theta' \in \Theta$ such that $\theta' \succsim_{FOSD} \theta$, we have $\Pi(q, t, \theta') \geq \Pi(q, t, \theta)$. In particular, $\Pi^*(\theta') \geq \Pi^*(\theta)$.*

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[3]I thank Juuso Toikka for suggesting this line of argument.

Proof. Part 1 follows immediately from the fact that $w^*(v)$ is non-decreasing: For every $v, v' \in V$ such that $v' \geq v$,

$$w^*(v') \geq v(q^{FB}(v), v') - c(q^{FB}(v)) \geq v(q^{FB}(v), v) - c(q^{FB}(v)) = w^*(v).$$

The first sentence in part 2 is an immediate consequence of FOSD. For the second sentence, note first that the constraint $q(v) \leq q^{FB}(v)$ for all $v \in V$ is non-binding in the principal's optimization problem. Thus, we can impose it on the principal's problem without loss of generality, and thereby invoke Lemma 1. (We abuse notation by continuing to denote by $\mathcal{M}_\theta, \mathcal{M}_{\theta'}$ the set of implementable mechanisms that satisfy the additional constraint.) If $V_\theta = V_{\theta'}$, then $\mathcal{M}_\theta = \mathcal{M}_{\theta'}$, and so:

$$\Pi^*(\theta') = \sup(\{\Pi(q, t, \theta') : (q, t) \in \mathcal{M}_{\theta'}\}) \geq \sup(\{\Pi(q, t, \theta) : (q, t) \in \mathcal{M}_\theta\}) = \Pi^*(\theta).$$

If $V_\theta \neq V_{\theta'}$, notice that FOSD implies (under full support) that $\underline{v}_{\theta'} \geq \underline{v}_\theta$ and $\overline{v}_{\theta'} \geq \overline{v}_\theta$. Construct the set of mechanisms where the implementability constraints are applied to all $v \in [\underline{v}_\theta, \overline{v}_{\theta'}]$; denote it by $\widetilde{\mathcal{M}}$. Thus,

$$\begin{aligned}
\Pi^*(\theta) &= \sup\left(\left\{\Pi(q, t, \theta) : (q, t) \in \widetilde{\mathcal{M}}\right\}\right) \quad \text{since } P_\theta([\overline{v}_\theta, \overline{v}_{\theta'}]) = 0; \\
&\leq \sup\left(\left\{\Pi(q, t, \theta') : (q, t) \in \widetilde{\mathcal{M}}\right\}\right) \quad \text{by Lemma 1;} \\
&= \Pi^*(\theta') \quad \text{since } P_{\theta'}([\underline{v}_\theta, \underline{v}_{\theta'}]) = 0.
\end{aligned}$$

This establishes the proposition. □

The core of the issue at hand is whether the distribution shift increases the social surplus by at least as much as the information rent. This is exactly what Lemma 1 establishes ex-post, while FOSD passes the result on to the expectations.

Notice that the argument involves $\Pi(q, t, \theta)$, which does not entail the typical substitution of transfers—and hence does not feature virtual utilities. While the characterization of incentive compatibility is independent of the type distribution (except for the support), information rents are not. If we did the substitution, the expression for ex-post payoffs would depend on the shape of the distribution trough virtual utilities; thus, FOSD would not (directly) apply.

Herein lies the power of the argument: By *not* substituting for the transfers in $\Pi(q, t, \theta)$, we can invoke FOSD—and we do not even assume that the distributions are regular, so the result follows *even if we need to iron the superior distribution*.

4   Linear Utilities and The Increasing Convex Order

In this section, we focus on the linear-utility setting $v(q, v) = vq$ and substitute for the transfers to further exploit the structure of the principal's problem. The goal is to find a "tighter" stochastic order that allows us to replicate the motonicity result from Proposition 1.

Under linear utilities, we can directly identify the agent's type with his valuation per unit. In this case, $q^{FB}(v) = (c')^{-1}(v)$ and $w^{*'}(v) = q^{FB}(v)$, which makes $w^*(v)$ non-decreasing and convex. Thus, monotonicity of $W^*(\theta)$ is now ensured under the increasing convex order.

Definition (Shaked and Shanthikumar, 2007). For any $\theta, \theta' \in \Theta$, θ' dominates θ in the *increasing convex order*, $\theta' \succsim_{ICxOD} \theta$, if for any non-decreasing and convex function $g : \mathbb{R} \to \mathbb{R}$, the expectation of $g(v)$ under θ' is at least as high as under θ:

$$\int_{V_{\theta'}} g(v) f_{\theta'}(v) \mathrm{d}v \geq \int_{V_{\theta'}} g(v) f_{\theta'}(v) \mathrm{d}v.$$

It is immediate to see that $\theta' \succsim_{FOSD} \theta$ implies $\theta' \succsim_{ICxOD} \theta$. Intuitively, if $\theta' \succsim_{ICxOD} \theta$, then the distribution identified by θ' has both a higher mean and a higher variance than its θ counterpart. Thus, the former has more mass on more extreme types in a way that raises the surplus. This order is the convex counterpart of second-order stochastic dominance. Shaked and Shanthikumar (2007) shows that $\theta' \succsim_{ICxOD} \theta$ if and only if:

$$\int_{V_{\theta'}} \max\{v - x, 0\} f_{\theta'}(v) \mathrm{d}v \geq \int_{V_\theta} \max\{v - x, 0\} f_\theta(v) \mathrm{d}v \tag{1}$$

for every $x \in \mathbb{R}$ (Theorem 4.A.2).

In the example from Section 2, it can be checked that $2 \succsim_{ICxOD} 1$, where $2 = \text{Beta}(4,4)$ and $1 = \text{Beta}(5,5)$.

Under asymmetric information, invoking incentive compatibility, we can write the principal's expected profit in terms of the allocation rule alone:

$$\Pi(q, \theta) = \int_{\underline{v}_\theta}^{\overline{v}_\theta} \left[u_\theta(v) \cdot q(v) - c(q(v)) \right] f_\theta(v) \mathrm{d}v,$$

where $u_\theta(v) = v - \frac{1 - F_\theta(v)}{f_\theta(v)}$ is the agent's virtual utility. This last expression highlights

the efficiency-rent trade-off that the principal faces: Generating more surplus can be profitable, but it can also mean more information rents for the agent.

A profit-maximizing principal will only serve types such that $u_\theta(v) \geq c'(0)$; other types end up shut down. Thus, we define the *truncated virtual utility*:

$$u_\theta^\tau(v) = \max \left\{ u_\theta(v), c'(0) \right\}. \tag{2}$$

For any given $\theta \in \Theta$, if $F_\theta(v)$ is *regular* (i.e., if $u_\theta(v)$ is non-decreasing), then the monotonicity constraint in the principal's maximization problem can be omitted.[4] In this case, we find the optimal allocation rule by maximizing pointwise, leading to:

$$q_\theta^*(v) = (c')^{-1} \left(u_\theta^\tau(v) \right) = q^{FB} \left(u_\theta^\tau(v) \right). \tag{3}$$

If the regularity assumption fails, the optimal allocation rule is found by replacing $u_\theta^\tau(v)$ in (3) with the ironed truncated virtual utility: $\bar{u}_\theta^\tau(v) = \max \{ \bar{u}_\theta(v), c'(0) \}$, where $\bar{u}_\theta(v)$ is the ironed virtual utility from Myerson (1981).

Examples 1 through 3 all fall in the category of linear utilities. Since they also feature $c'(q) = q$, we have that $q_\theta^*(v) = u_\theta^\tau(v)$—which is non-decreasing in all three cases. Moreover, as noted above, all of these examples involve a shift in the sense of FOSD.

To determine whether we can find a weaker condition, let $\tilde{u}_\theta^\tau(v)$ equal $u_\theta^\tau(v)$ if θ is regular and $\bar{u}_\theta^\tau(v)$ otherwise; the principal's maximum expected profit is:

$$
\begin{aligned}
\Pi^*(\theta) &= \int_{\underline{v}_\theta}^{\overline{v}_\theta} \left[\tilde{u}_\theta^\tau(v) \cdot q_\theta^*(v) - c(q_\theta^*(v)) \right] f_\theta(v) \mathrm{d}v \\
&= \int_{\underline{v}_\theta}^{\overline{v}_\theta} \left[\tilde{u}_\theta^\tau(v) \cdot q^{FB} \left(\tilde{u}_\theta^\tau(v) \right) - c \left(q^{FB} \left(\tilde{u}_\theta^\tau(v) \right) \right) \right] f_\theta(v) \mathrm{d}v \\
&= \int_{\underline{v}_\theta}^{\overline{v}_\theta} w^* \left(\tilde{u}_\theta^\tau(v) \right) f_\theta(v) \mathrm{d}v.
\end{aligned}
$$

The last equality suggests the following concept as the "minimal" condition for monotonicity of $\Pi^*(\theta)$, based on the analysis of the first-best benchmark:

Definition. For any $\theta, \theta' \in \Theta$, we say that θ' dominates θ in the *incentive dominance* (ID), $\theta' \succsim_{ID} \theta$, if $\tilde{u}_{\theta'}^\tau(v) \succsim_{ICxOD} \tilde{u}_\theta^\tau(v)$.

As providing incentives leads to replacing actual valuations with virtual utilities,

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[4]Notice that non-motonicities of $v - \frac{1 - F_\theta(v)}{f_\theta(v)}$ below $c'(0)$ are irrelevant.

the incentive order replicates the distribution ranking for social surplus but applied to the latter. Notice that ICxOD on the valuations is insuficient.

Example 1 (Continued). For each $\theta \in [1,2]$, we have $u_\theta(v) = \max\{2v - \theta, 0\}$. Thus, the distribution of $u_\theta(v)$ is given by the cdf $G_\theta(u) = \frac{\theta + u}{2\theta}$ with support $[0, \theta]$. This distribution has an atom of 0.5 at $u = 0$ and has the remaining mass uniformly distributed. Notice that $\theta' > \theta$ (so that $\theta' \succsim_{FOSD} \theta$) implies that $u_{\theta'}(v) \succsim_{FOSD} u_\theta(v)$, hence $\theta' \succsim_{ID} \theta$. $\triangle$

Example 3 (Continued). Recall that $u_\theta(v) = \max\{\theta v - \theta + 1, 0\}$ for $\theta \in (1, +\infty)$. The distribution of $u_\theta(v)$ is now given by the cdf:

$$G_\theta(u) = 1 - \left(\frac{1-u}{\theta}\right)^{\frac{1}{\theta - 1}}$$

with support $[0, 1]$. Figure 4 depics the cdfs for $\theta = 2$ (blue curve) and $\theta = 3$ (red curve). Once again, $\theta' > \theta$ (so that $\theta' \succsim_{FOSD} \theta$) implies that $u_{\theta'}(v) \succsim_{FOSD} u_\theta(v)$, hence $\theta' \succsim_{ID} \theta$. $\triangle$
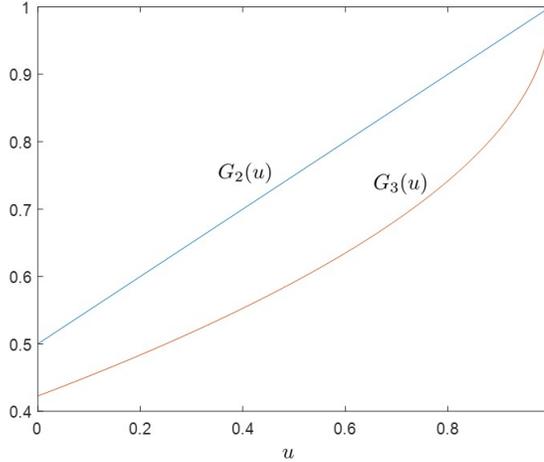

Figure 4: Cdfs for $u_\theta(v)$ in Example 3 for $\theta = 2$ (blue curve) and $\theta = 3$ (red curve).

The type distributions in these examples are ranked under FOSD, which the next proposition shows is a sufficient condition for ID.

To prove the proposition, it is convenient to change variables following Bulow and Roberts (1989): $x := 1 - F_\theta(v)$, $v_\theta(x) := F_\theta^{-1}(1 - x)$. Notice that x has a uniform distribution on $[0, 1]$ and that $v_\theta(x)$ is non-increasing. Next, define $R_\theta(x) := x \cdot v_\theta(x)$. This expression is analogous to a monopoly revenue formula, with x playing the role of quantity and $v_\theta(x)$ playing the role of inverse demand. We have that $R_\theta'(x) = u_\theta(v_\theta(x))$ and, defining $z(y) := \max\{y, c'(0)\}$, $\Pi^*(\theta) = \int_0^1 (w^* \circ z)\left(R_\theta'(x)\right) dx$ if θ is regular. Otherwise, Myerson's process of ironing entails replacing $R_\theta(x)$ with its concave envelope, $\bar{R}_\theta(x)$:

$$\bar{R}_\theta(x) := \inf\left(\{r(x) : r : [0, 1] \to \mathbb{R} \text{ is concave and, } \forall \tilde{x} \in [0, 1] \ (r(\tilde{x}) \geq R_\theta(\tilde{x}))\}\right),$$

which (by concavity) is differentiable almost everywhere.

Proposition 2. *For every $\theta, \theta' \in \Theta$, $\theta' \succsim_{FOSD} \theta$ implies $\theta' \succsim_{ID} \theta$.*

Proof. We start with the case in which both distributions are regular. Pick any $\theta, \theta' \in \Theta$ such that $\theta' \succsim_{FOSD} \theta$. The latter is equivalent to quantile dominance, which in this setting can be written as $v_{\theta'}(x) \geq v_\theta(x)$, hence $R_{\theta'}(x) \geq R_\theta(x)$, for all $x \in [0, 1]$. By regularity, $R_{\theta'}(x)$ and $R_\theta(x)$ are concave, hence differentiable almost everywhere and—where defined—their derivatives are non-increasing. Morover, since $R_{\theta'}(0) = R_\theta(0) = 0$,

$$\int_0^x R_{\theta'}'(y) dy = R_{\theta'}(x) \geq R_\theta(x) = \int_0^x R_\theta'(y) dy \tag{4}$$

for all $x \in [0.1]$. As $R_\theta'(y)$, $R_{\theta'}'(y)$ are non-increasing, they equal their decreasing rearrangements. Hence, inequality (4) means that $R_{\theta'}'$ weakly submajorizes R_θ' (Hardy, Littlewood, and Pólya, 1952; Marshall, Olkin, and Arnold, 2011). An extension of the Hardy-Littlewood-Pólya inequality from Chong (1974) (Theorem 2.3) states that weak submajorization is equivalent to the following:

$$\int_0^1 h\left(R_{\theta'}'(x)\right) dx \geq \int_0^1 h\left(R_\theta'(x)\right) dx$$

for all non-decreasing and convex functions $h : \mathbb{R} \to \mathbb{R}$. The result follows by noting that, if $g : \mathbb{R} \to \mathbb{R}$ is one such non-decreasing and convex function, so is $(g \circ z)$, and

thus:

$$\int_{V_\theta} g(u_\theta(v)f_\theta(v)\mathrm{d}v = \int_0^1 (g \circ z)\left(R'_\theta(x)\right)\mathrm{d}x$$

$$\leq \int_0^1 (g \circ z)\left(R'_{\theta'}(x)\right)\mathrm{d}x = \int_{V_{\theta'}} g(u_{\theta'}(v)f_{\theta'}(v)\mathrm{d}v.$$

If one or both of the distributions is not regular, the corresponding virtual utility must be ironed. We start with the case where θ is regular but θ' is irregular. We have that $\bar{R}_{\theta'}(x) \geq R_{\theta'}(x) \geq R_\theta(x)$ for all $x \in [0,1]$. Since $R_{\theta'}(x)$ is concave, it is differentiable almost everywhere and—along with the fact that $\bar{R}_\theta(0) = 0$—we have that $\bar{R}_{\theta'}(x) = \int_0^x \bar{R}'_{\theta'}(y)\mathrm{d}y$. Thus, $\bar{R}_{\theta'}(x)$ weakly submajorizes R_θ, and the rest of the argument follows as above. If θ is irregular but θ' is regular, weak submajorization follows from the observation that $R_{\theta'}(x)$ is in the set that defines $\bar{R}_\theta(x)$, so $R_{\theta'}(x) \geq \bar{R}_\theta(x)$ for all $x \in [0,1]$. Finally, combining the two cases tackles the case when both θ and θ' are irregular.

Alternatively, following Roughgarden and Schrijvers (2015), ironing the virtual utility is equivalent to replacing the irregular distribution $F_{\theta'}$ with a regular one $\bar{F}_{\theta'}$ such that: (1) $\bar{F}_{\theta'} \succsim_{FOSD} F_{\theta'}$; (2) The ironed virtual utility for $F_{\theta'}$ is identical to the virtual utility for $\bar{F}_{\theta'}$; and (3) The expected profit given the ironed allocation rule under $F_{\theta'}$ and $\bar{F}_{\theta'}$ are equal. The Lemma in the appendix shows that, for any regular type distribution G, $F_{\theta'} \succsim_{FOSD} G$ implies $\bar{F}_{\theta'} \succsim_{FOSD} G$ and $G \succsim_{FOSD} F_{\theta'}$ implies $G \succsim_{FOSD} \bar{F}_{\theta'}$. Thus, the previous argument applies, and it can be easily adapted to when F_θ is the irregular distribution, or when both are irregular. $\square$

While sufficient, FOSD is not a necessary condition for dominance under ID, as the next example shows.

Example 4. Consider two distributions $\theta = 1, 2$ with support $[0,1]$ constructed from a hazard function $h_\theta(v) := \frac{f_\theta(v)}{1-F_\theta(v)} = \frac{1}{\alpha_\theta(v)(1-v)}$, where $\alpha_\theta(v)$ is piecewise linear. Figure 5 depicts the cdfs $F_1(v)$ and $F_2(v)$, showing that the two distributions are not comparable under FOSD. Take $c'(0) = 0$; to verify that $2 \succsim_{IO} 1$, we construct the virtual utilities and apply condition 1; see Figure 6. $\triangle$

While not equivalent, the proof of Proposition 2 suggests that FOSD and ID are closely connected. In fact, the next proposition shows that FOSD is equivalent to ICxOD applied to the "raw" virtual utilities on the domain of regular distributions.
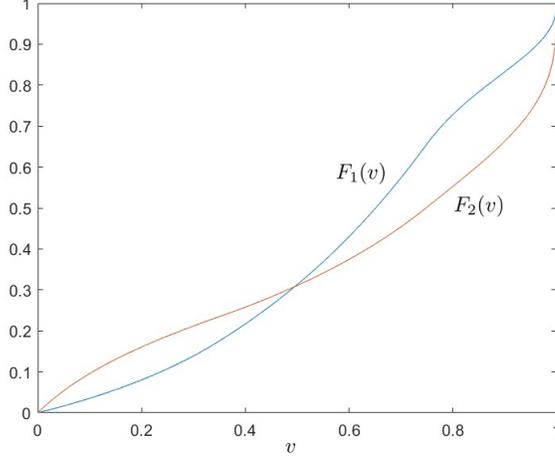
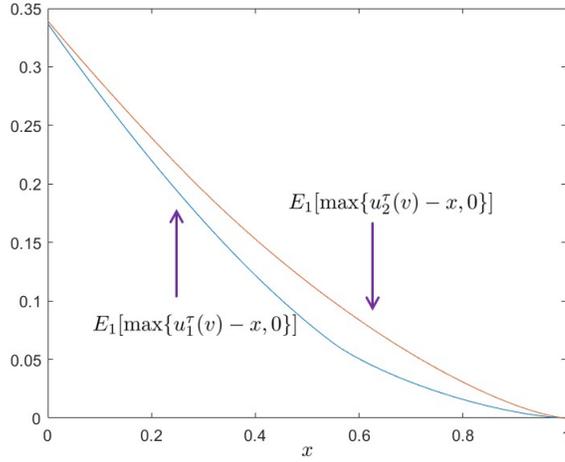Figure 5: Graphs of $F_1(v)$, $F_2(v)$ in Example 4.

Figure 6: Verifying that $2 \succsim_{ID} 1$ in Example 4.

Proposition 3. *For every $\theta, \theta' \in \Theta$, $\theta' \succsim_{FOSD} \theta$ implies $\tilde{u}_{\theta'}(v) \succsim_{ICxOD} \tilde{u}_{\theta'}(v)$. Conversely, if θ, θ' are regular, $u_{\theta'}(v) \succsim_{ICxOD} u_{\theta'}(v)$ implies $\theta' \succsim_{FOSD} \theta$.*

Proof. The ($\Rightarrow$) direction is essentially the proof of Proposition 2, so we omit further details. As for the ($\Leftarrow$) direction, the missing piece is to establish that "revenue dominance" $R_{\theta'}(x) \geq R_\theta(x)$ for every $x \in [0,1]$ implies that $\theta' \succsim_{FOSD} \theta$. Under regularity, $R_\theta(x) = x \cdot F_\theta^{-1}(1-x)$ and $R_{\theta'}(x) = x \cdot F_{\theta'}^{-1}(1-x)$. Thus, the inequality reduces to $F_{\theta'}^{-1}(p) \geq F_\theta^{-1}(p)$ for every $p \in (0,1)$, namely $\theta' \succsim_{FOSD} \theta$. □

Thus, for regular distributions $\theta, \theta' \in \Theta$, ICxOD applied to $u_\theta(v)$, $u_{\theta'}(v)$ is an alternative characterization of FOSD.

If at least one of the distributions is not regular, its virtual utility must be replaced with its ironed virtual utility. The latter corresponds to the (derivative of the) concave envelope of the revenue characterization rather than the revenue itself, breaking the tight link between revenue and quantiles.

Example 5. Let $\theta = 1$ represent the uniform distribution on $[0, 1]$, while $\theta = 2$ is the Beta$(0.7, 0.3)$ distribution. Figure 7 shows that $2 \not\succeq_{FOSD} 1$. Figure 8 shows that $R_2(x) < R_1(x)$ for sufficiently high $x < 1$, and yet $\bar{R}_2(x) \geq R_1(x)$. △
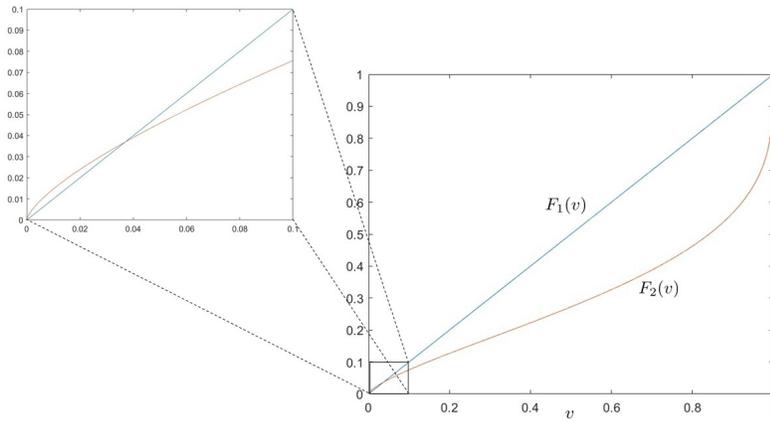

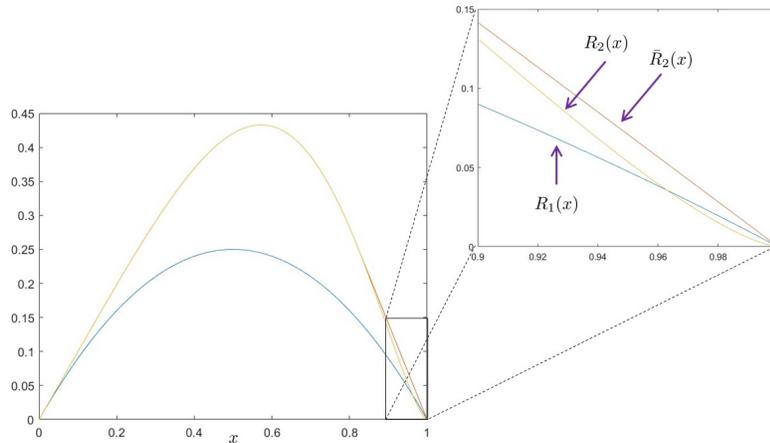Figure 7: Graphs of $F_1(v)$, $F_2(v)$ in Example 5.

Figure 8: Graphs of $R_1(x)$, $R_2(x)$, $\bar{R}_2(x)$ in Example 5.

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5 Conclusion

This paper addresses the question of when a "social improvement" in the agent's type distribution is also "privately" beneficial for a principal who trades off social surplus for lower information rents. As with externalities, information rents mean that socially-beneficial shifts in the distribution of agent types may lead to a drop in the principal's expected payoff.

In a quasilinear setting, first-order stochastic dominance is a sufficient condition for social and private improvements to align—namely, for both maximum expected social surplus and maximum expected profit to be non-decreasing.

Looking for a weaker condition, we further focus on the case of linear utilities, where dominance in the increasing convex order corresponds to social improvements. Since maximum expected profit can be characterized as maximum expected social surplus with the agent's types replaced with the possibly-truncated, possibly-ironed virtual utilties, we propose *incentive dominance*, which is dominance in the increasing convex order applied to the latter.

We show that incentive dominance is indeed weaker than first-order stochastic dominance. However, we also show that it is not "that much weaker": For regular distributions, first-order stochastic dominance is equivalent to dominance in the increasing convex order applied to non-truncated virtual utilities.

In the paper, the principal plays the role of a monopolist and the agent is a buyer with privately-known valuations drawn from a continuum. The results can be reformulated for a setting with discrete types, invoking majorization for vectors. In a procurement setting where the agent is a contractor with privately-known costs, the corresponding notion of incentive dominance is based on the increasing concave order, a.k.a. second-order stochastic dominance.

A Appendix

The lemma below establishes some properties of the "ironed" distribution from Roughgarden and Schrijvers (2015). For the sake of completeness, we include the construction of said distribution and some of its basic properties expressed in our setting.

Lemma (Ironed distributions)**.** *Let F be the cdf of an absolutely continuous and full-support distribution on an interval $[\underline{v}, \overline{v}]$. There exists an absolutely continuous, full-support,*

and regular distribution $\bar{F}$ on $[\underline{v}, \overline{v}]$ such that:

1. *$\bar{F} \succsim_{FOSD} F$ and $\bar{F} = F$ outside of the bunching interval;*

2. *The ironed virtual utility for F, $\overline{u}_F(v)$, equals the virtual utility for $\bar{F}$, $u_{\bar{F}}(v)$;*

3. *$\int_0^1 (h \circ z) \left(\bar{R}'(x) \right) dx = \int_0^1 (h \circ z) \left(R'_{\bar{F}}(x) \right) dx;$*

4. *For any regular type distribution G, $F \succsim_{FOSD} G$ implies $\bar{F} \succsim_{FOSD} G$;*

5. *For any regular type distribution G, $G \succsim_{FOSD} F$ implies $G \succsim_{FOSD} \bar{F}$.*

Proof. Take $R(x)$ and its concave upper envelope, $\bar{R}(x)$. Define $\overline{v}(x) : [0, 1] \to [\underline{v}, \overline{v}]$ as $\overline{v}(x) = \frac{\bar{R}(x)}{x}$ for $x > 0$ and $\overline{v}(0) := \lim_{x \to 0^+} \frac{\bar{R}(x)}{x}$. Since $\bar{R}(x) \geq R(x)$, we have that $\overline{v}(x) \geq v(x)$, with equality outside the bunching interval $[x_1, x_2]$, where $x_1 = 1 - F(v_1)$ and $x_2 = 1 - F(v_2)$. Moreover, $\overline{v}(x)$ is of the form $m + \frac{b}{x}$—since $\bar{R}(x)$ is affine—on the bunching interval, where $b > 0$ and where $m \geq 0$ is the ironed value of the ironed virtual utility. Define $\bar{F}$ as the cdf whose quantiles correspond to $\overline{v}(x)$:

$$\bar{F}(v) = 1 - \inf\left(\{x \in [0, 1] : v \geq \overline{v}(x)\}\right).$$

To show 1, notice that the inequality $\overline{v}(x) \geq v(x)$ implies $\{x \in [0, 1] : v \geq \overline{v}(x)\} \subseteq \{x \in [0, 1] : v \geq v(x)\}$, and so:

$$\inf\left(\{x \in [0, 1] : v \geq \overline{v}(x)\}\right) \geq \inf\left(\{x \in [0, 1] : v \geq \overline{v}(x)\}\right) = 1 - F(v).$$

So, $\bar{F}(v) \leq 1 - (1 - F(v)) = F(v)$. Next, take any v outside of the bunching interval; for the corresponding $x = 1 - F(v)$, we have that $\overline{v}(x) = v(x)$, so the result follows.

To establish 2, we need only to look at the bunching interval, where $\overline{v}(x) = m + \frac{b}{x}$. Thus, the inequality $v \geq \overline{v}(x)$ can be rewritten as $x \geq \frac{b}{v-m}$, since $v > m$ on the bunching interval (due to downward distortion). Thus, $\bar{F}(v) = 1 - \frac{b}{v-m}$.[5] Straightforward computation shows that $u_{\bar{F}}(v) = m = \overline{u}_F(v)$.

Part 3 is a special case of a result in Myerson (1981), although in the present context it is an immediate consequence of the fact that $\bar{R}'(x) = R'(x)$ for every $x \in [0, 1]$.

For 4, let G be the cdf of a regular distribution that $G \succsim_{FOSD} F$. Regularity implies that $R_G(x)$ is concave, while dominance implies that $R_G(x) \geq R(x)$ for every $x \in [0, 1]$. Thus, $R_G(x) \geq \bar{R}(x)$, which implies that $G \succsim_{FOSD} \bar{F}$.

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[5] Alternatively, this expression for $\bar{F}$ on the bunching interval can be obtained as the solution to the ODE $\frac{1-\bar{F}(v)}{\bar{f}(v)} = v - m$ with boundary conditions $\bar{F}(v_1) = F(v_1)$ and $\bar{F}(v_2) = F(v_2)$.

For 5, let G be the cdf of a regular distribution such that $F \succsim_{FOSD} G$. Then, $R(x) \geq R_G(x)$ for every $x \in [0,1]$, so $\bar{R}(x) \geq R_G(x)$ follows immediately from 1. $\square$

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