

DISCRETE SCREENING*

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Abstract

We consider a principal who wishes to screen an agent with *discrete* types by offering a menu of *discrete* quantities and *discrete* transfers. We assume that the principal’s valuation is discrete strictly concave and use a discrete first-order approach. We model the agent’s cost types as non-integer, with integer types as a limit case. Our modeling of cost types allows us to replicate the typical constraint-simplification results and thus to emulate the well-treaded steps of screening under a continuum of contracts.

We show that the solutions to the discrete F.O.C.s need not be unique *even under discrete strict concavity*, but we also show that there cannot be more than two optimal contract quantities for each type, and that—if there are two—they must be adjacent. Moreover, we can only ensure weak monotonicity of the quantities *even if virtual costs are strictly monotone*, unless we limit the “degree of concavity” of the principal’s utility. Our discrete screening approach facilitates the use of rationalizability to solve the screening problem. We introduce a rationalizability notion featuring robustness with respect to an open set of beliefs over types called Δ -O *Rationalizability*, and show that the set of Δ -O rationalizable menus coincides with the set of usual optimal contracts—possibly augmented to include irrelevant contracts.

Keywords: Screening; discrete concave optimization; rationalizability; level- k reasoning.

JEL Classification Numbers: D82.

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1 Introduction

We consider a principal (she) who wants to procure a good from an agent (he). The agent’s cost structure, represented by his marginal cost, is his private information. In order to screen the agent, the principal offers a menu of contracts, each contract specifying a quantity and monetary transfer catered to each of the agent’s possible marginal cost types (Mirrlees, 1971; Mussa and Rosen, 1978; Baron and Myerson, 1982; Maskin and Riley, 1984).

In contrast to the existing literature, we focus on the case where marginal cost types, quantities, and transfers are *all* discrete. There are at least three reasons for it: First, this case is more realistic. Quantities can only be measured with finite precision, and monetary transfers have smallest units. While the standard first-order approach used to study the screening problem is elegant and extremely useful, it is important to scrutinize whether the well-known results of continuous marginal analysis of the screening problem can be regained in a entirely discrete setting. Second, the discrete approach to screening facilitates going beyond the equilibrium analysis. For instance, rationalizability (Bernheim, 1984; Pearce, 1984) is a desirable solution concept when no equilibrium behavior can be assumed such as when the principal is *unaware* of some of the agent’s types, as we analyze in Francetich and Schipper (2025). Our analysis here provides the full-awareness benchmark to Francetich and Schipper (2025). Compared to standard approaches to screening, we give up the common prior assumption. Third, although we are not exploring this here further, classical computability based on the Church-Turing Hypothesis fundamentally requires discreteness. Our approach should be instrumental in facilitating the analysis of computability of the screening problem.

There are two important challenges when dealing with discrete contracts and when taking a rationalizability approach, respectively. With discrete contracts, even if some of the usual constraint-simplification results can be extended, the characterization of transfers in terms of surplus and information rents may not hold. This could preclude substituting for the transfers in the principal’s objective function, further complicating the problem. Under rationalizability, in addition, the usual equilibrium tie-breaking assumptions to resolve the agent’s indifferences between different contracts and/or the outside option cannot be assumed.

To expand on the last point: When no equilibrium tie-breaking convention can be assumed, types of the agent who are indifferent between different contracts in the menu, or between the menu and their outside option, may choose the “wrong” contract and create unintended bunching. To counter, the principal may want to provide strict incentives for the agent to self-select into the “right” choice, i.e., impose incentive compatibility and participation constraints with strict inequalities. With a continuum of contracts, the problem is that there is no well-defined “smallest transfer” to break the indifference implying non-existence of optimal contracts that observe strict incentive compatibility and participation constraints. However, discreteness by itself does not entirely avert the problem: If the transfer discrete concave screening can match the agent’s total cost or rent, the relevant incentive compatibility or participation constraints would still hold with equality.

We tackle all of these problems simultaneously through our specification of types as *non-integer* costs. Our specification of non-integer types introduces a *round-up rent* that makes all incentives strict, simplifying the rationalizability analysis and yielding sharper predictions. It also allows us to replicate the usual constraint-simplification results and characterization of transfers for discrete contracts, facilitating the comparison of our results with those obtained

from standard equilibrium screening. It features the case of integer costs as a limiting case. Aside from the technical convenience, non-integer types can reflect the fact that private costs can be nuanced, idiosyncratic, implicit, unverifiable and even cognitive and subjective while the terms of a contract are typically standardized by objective scales of measurements that contracts can explicitly be written on and that can be verified ex post. We believe that’s one of the reasons why contracts cannot directly conditions on marginal costs. We also note that if the discrete concave screening for quantities and transfers and the discrete concave screening for marginal costs are independently drawn at random from real-valued discrete concave screenings, then drawing the same discrete concave screening for both quantities and transfers on one hand and marginal costs on the other would be highly non-generic.

To solve the screening problem, we emulate a first-order approach using discrete forward and backward derivatives. Unlike in the standard continuum case, the optimal menu of contracts may not be unique *even under strict concavity* of the principal’s valuation function. Nonetheless, we show that there cannot be more than two optimal quantities for each type, and—if there are two—they must be adjacent. Moreover, this is a knife-edge result and can be avoided with a slight perturbation of the type distribution. Now, even if the principal is indifferent between offering two different quantities to a given type, the agent is not: We show that the agent always prefers the larger quantity, as the corresponding contract yields a higher round-up rent.

Another departure from standard screening is that different types may be awarded the same quantity in the optimal contract *even if virtual costs are strictly monotone*. In other words, we may have “non-binding” bunching, in the sense that the solution to the principal’s problem omitting the monotonicity constraints satisfies at least some of the latter with equality. This can happen if the principal’s valuation function is “too concave,” so we introduce a condition “limiting” its concavity that ensures strict monotonicity.

Having characterized the optimal discrete screening program of the principal, we develop a rationalizability approach. Rationalizability is an iterative reduction procedure on beliefs and strategies. At the first level, each player keeps all strategies that are rational w.r.t. some belief. For the agent, the beliefs over the principal’s strategy are degenerate because he observes the menu of contracts offered by the principal. Thus, for the agent, rationality means simply self-selecting into the best contract in the menu or the outside option—effectively satisfying incentive compatibility constraints and participation constraints. The principal, in contrast, must form beliefs not just about the agent’s strategy but also about his type.

The first obstacle to a rationalizability analysis is that not all marginal beliefs on types are consistent with monotone virtual costs. Thus, we restrict the principal’s marginal beliefs over types to be log-concave, a standard assumption in information economics (e.g., Bagnoli and Bergstrom, 2005) interpreted as the principal having single-peaked beliefs over the agent’s types. Rationalizability notions with belief restrictions in games with incomplete information have been studied at least since Battigalli and Siniscalchi (2003) and Battigalli (2003) under the name of Δ -Rationalizability (see Battigalli and Siniscalchi, 2003; Battigalli and Friedenberg, 2012; Battigalli and Prestipino, 2013; Brandenburger, Friedenberg, and Keisler, 2008; Brandenburger and Friedenberg, 2010). However, we are not aware that the restriction to log-concavity has been explored previously in rationalizability notions.

The second obstacle, as noted earlier, is that discrete strict concavity does not imply uniqueness of optimal contracts. As this is a knife-edge case with respect to the principal’s marginal beliefs on marginal cost types, we rule out non-uniqueness by requiring robustness to belief

perturbations. More precisely, we require the principal’s strategy to be rational w.r.t. an open set of beliefs over marginal cost types.¹ Hence, we call our rationalizability concept Δ -O *Rationalizability*.

As usual for rationalizability notions, at higher levels ℓ , players believe in $(\ell - 1)$ level strategies of the opponent if possible. For the principal, this means that at level 2 she is certain that the agent observes incentive compatibility and participation constraints. Thus, she essentially solves her program at the second level; no further reductions of the set of strategies occur at levels higher than 2. Moreover, we show that the set of Δ -O rationalizable contracts coincides with the set of usual optimal menus (for the corresponding restricted beliefs), possibly expanded to include redundant contracts.

We are not aware of any rationalizability concept that require a rationalizable strategy to be best response to all beliefs in a nonempty open set of beliefs. Perhaps the closest approach (for static games with complete information) is by Ziegler and Zuazo-Garin (2020) who require a rationalizable strategy to be a best response to all beliefs in a subset of beliefs over remaining strategies. Their aim is to model cautiousness and ambiguity. In contrast, we use the idea in the context of incomplete information to model robustness to small perturbations of beliefs over marginal-cost types and hence slight perturbations of virtual costs.

Our paper contributes to the classic literature on screening. Both the standard case with a continuum of types or two discrete types and continuum of quantities and transfers have been extensively studied and applied throughout economics. The case with a finite number of types but a continuum of quantities and transfers has also been studied in textbooks (e.g., three marginal cost-types in Laffont and Martimort, 2001). Yet, we are not aware of any study using discrete contracts, even though it is very realistic. We also contribute to the literature on rationalizability in games with incomplete information (Battigalli, 2003, 2006; Battigalli and Siniscalchi, 2003; Heifetz, Meier, and Schipper, 2021; Li and Schipper, 2019) by exploring rationalizability with logconcavity belief-restrictions on marginal cost-types and robustness via rationality with respect to an open set of beliefs.

In Francetich and Schipper (2025), we expand on the problem by making the principal initially *unaware* of some of the agent’s types. The agent has the option to raise the principal’s awareness, totally or partially, before contracting. Further restricting beliefs to satisfy reverse Bayesianism, a “wariness” condition, and a monotonicity condition, we show that if the principal is ex ante only unaware of high cost types, all of these types have an incentive raise her awareness of them—otherwise, they would not be served. In the special case of three types, the two lower cost types that the principal is initially aware of also want to raise her awareness of the high cost type; their quantities suffer no additional distortions and they both earn an extra information rent. This is very intuitive: The presence of an even higher cost type makes the original two look better.

However, with more than three types, we show that this intuition may break down for types of whom the principal is initially aware of so that raising the principal’s awareness could cease to be profitable for those types. The reason is that after raising awareness of higher-cost types, the principal may focus her belief on higher-cost types. Consequently, the principal may not design a menu with a dedicated contract for such lower-cost type, who would then need to

¹As the type space is finite, the set of distributions is identified with a simplex and we employ the usual topology.

bunch and select a contract dedicated for another type, which might be less appealing than the contract selected by the type without raising awareness. When the principal is ex ante only unaware of more efficient (i.e., lower-cost) types, then *no type* raises her awareness, leaving her none the wiser. Intuitively, raising awareness of lower-cost types makes others look worse in the eye of the principal, while said types can enjoy a higher surplus under the “default” offer.

The paper is organized as follows. Section 2 presents the model. Section 3 analyzes the principal’s screening problem and shows that the usual solution may fail to be unique and strictly monotone. Section 4 further characterizes optimal contracts, while Section 5 develops our rationalizability analysis. Finally, Section 6 concludes. Appendix A presents tools and results of discrete concave optimization, while Appendix B features characterizations of log-concavity of beliefs on types.

2 Model

A principal (P , “she”) wants to procure a quantity q of an object from an agent (A , “he”) in exchange for a transfer t . We assume that $q, t \in D := \{0, 1, \dots, b\}$ for some $b \in \mathbb{N}$. The agent’s marginal cost is his private information; we denote it by $\theta \in \Theta$, where:

$$\Theta = \left\{ 1 - \frac{1}{\gamma}, \dots, m - \frac{1}{\gamma} \right\}$$

for some $\gamma > b > m$. We require that both γ and b are large compared to m , so that the principal is able to design contract menus with as many contracts as cost types there are—if she desires to do so.²

Notice that the types in Θ are non-integer. Our specification of non-integer types allows us to accomplish two simultaneous goals: (1) To replicate the usual constraint-simplification steps and thus obtain formulas more-closely comparable to standard results; and (2) to break indifferences in incentive and participation constraints due to the mismatch between integer transfers and non-integer total costs. Moreover, we can accommodate integer types by simply taking limit as $\gamma \rightarrow \infty$. Of course, in the limit, we revert back to possibly weak incentive and participation constraints. If we take the usual equilibrium screening approach, resolving indifferences in favor of the principal, this is not a problem.

The principal screens the agent by offering him the choice of a contract $\mathbf{c} := (q, t)$ from a menu of contracts M . The set of contract menus is $\mathcal{M} := 2^{D^2} \setminus \{\emptyset\}$. Presented with a menu of contracts $M \in \mathcal{M}$, the agent chooses either a contract $\mathbf{c} = (q, t) \in M$ or takes his outside option, $\mathbf{o}$, identified by the null-contract $(0, 0)$, and the game ends. The payoff for the agent with type θ from signing contract $\mathbf{c} = (q, t)$ is given by $u_A(\mathbf{c}, \theta) := t - \theta q$, while he earns $u_A(\mathbf{o}, \theta) = 0$ from the outside option. The principal’s benefit of output is given by a function $v : D \rightarrow \mathbb{R}$, and her utility or payoff from a contract $\mathbf{c} = (q, t)$ is $u_P(\mathbf{c}) = v(q) - t$. We assume the following about $v(q)$.

Assumption 1 *The function v satisfies the following properties:*

²Although our representation of cost types is unidimensional, we can think of $\theta \in \Theta$ as a score that aggregates the impact of various factors affecting marginal costs, reminiscent of “pseudotypes” in scoring auctions; see, for instance, Asker and Cantillon (2006) or Bajari, Houghton, and Tadelis (2014).

1. $v(0) = 0$;
2. $v(q)$ is strictly increasing in q ;
3. $v(q)$ is strictly discrete concave in q : For every $q \in \{1, \dots, b-1\}$, $v(q+1) + v(q-1) < 2v(q)$.

Assumptions 1.1 and 1.2 are standard in the screening literature, while Assumption 1.3 is the discrete counterpart of the typical assumption of strict concavity in the literature.

We can characterize the second and third conditions in terms of *discrete derivatives*. Define $\Delta^-v(q) := v(q) - v(q-1)$ for $q \in \{1, \dots, b\}$ and $\Delta^+v(q) := v(q+1) - v(q)$ for $q \in \{0, \dots, b-1\}$; these are the discrete backward and forward derivatives of $v(q)$, respectively. By Lemma A2 in the appendix, Assumption 1.2 is equivalent to either one of the conditions $\Delta^-v(q) > 0$ for $q \in \{1, \dots, b\}$ or $\Delta^+v(q) > 0$ for $q \in \{0, \dots, b-1\}$. Defining the second discrete derivative $\Delta^+\Delta^-v(q) := \Delta^+(\Delta^-v(q))$, we can restate Assumption 1.3 as $\Delta^+\Delta^-v(q) < 0$ for $q \in \{1, \dots, b-1\}$.

The principal chooses a menu of contracts to maximize her expected payoff under some full-support belief about types, $p \in \text{int}(\Delta(\Theta))$ with $\Delta(\Theta)$ denoting the set of probability measures on Θ and $\text{int}(\Delta(\Theta))$ being the interior of $\Delta(\Theta)$. Note that p does not need to be a common prior. It could be any marginal belief over cost types that the principal uses to rationalize a menu of contract.

3 The Screening Problem

In this section, we set up the screening problem and employ the structure of our marginal cost type space to obtain constraint-simplification results that mimic the traditional ones. To facilitate the analysis, we define recursively the j -th highest order statistics of the marginal cost types for $j \in \{1, \dots, m\}$, respectively, by

$$\kappa^{(1)} := \max \Theta, \quad \text{and} \quad \kappa^{(j)} := \max(\Theta \setminus \{\kappa^i : i = 1, \dots, j-1\}) \quad (1)$$

for $j = 2, \dots, m$. Denote by (q^i, t^i) the contract designed for type $\kappa^{(i)}$, and by p^i the probability of $\kappa^{(i)}$ under p , for $i = 1, \dots, m$. We can write the principal's optimization problem as:

$$\max_{(q^i, t^i)_{i=1, \dots, m} \in (D^2)^m} \sum_{i=1}^m p^i (v(q^i) - t^i)$$

subject to the usual incentive compatibility and participation constraints, IC and PC, respectively: For all $i, j = 1, \dots, m$:

IC _{i,j} :

$$t^i - \kappa^{(i)} q^i \geq t^j - \kappa^{(i)} q^j$$

PC _{i} :

$$t^i - \kappa^{(i)} q^i \geq 0.$$

We will show that several of the standard constraint-simplification results can be extended to our discrete setting. For instance, PC_1 implies PC_i for every other i under incentive compatibility, and the only relevant incentive constraints are the local ones if the allocation rule is monotone.

Lemma 1 *For all $i = 2, \dots, m$, PC_1 and $IC_{i,i-1}$ implies PC_i .*

PROOF. Observe that:

$$0 \leq t^1 - \kappa^{(1)} q^1 \stackrel{\kappa^{(1)} > \kappa^{(2)}}{<} t^1 - \kappa^{(2)} q^1 \stackrel{IC_{2,1}}{\leq} t^2 - \kappa^{(2)} q^2 \\ \stackrel{\kappa^{(2)} > \kappa^{(3)}}{<} t^2 - \kappa^{(3)} q^2 \leq \dots \stackrel{IC_{m,m-1}}{\leq} t^m - \kappa^{(m)} q^m.$$

This establishes the result. $\square$

Lemma 2 *If for all $i = 2, \dots, m$, $q^i \geq q^{i-1}$ ($q^i > q^{i-1}$), then $IC_{i,i-1}$ implies $IC_{i,j}$ (with strict inequality) for all $j < i$.*

The proof is analogous to the proof of the next lemma:

Lemma 3 *If for all $i = 1, \dots, m-1$, $q^i \geq q^{i-1}$ ($q^i > q^{i-1}$), then $IC_{i,i+1}$ implies $IC_{i,j}$ (with strict inequality) for all j with $m \geq j > i$.*

PROOF. We prove by induction on the index of the order statistics. The base case is just $IC_{i,i+1}$.

Induction hypothesis: For some $i, j = 1, \dots, n-1$ with $i+j < m$,

$$t^i - \kappa^{(i)} q^i \geq (>) t^{i+j} - \kappa^{(i)} q^{i+j}.$$

Inductive step: We need to show:

$$t^{i+j} - \kappa^{(i)} q^{i+j} \geq (>) t^{i+j+1} - \kappa^{(i)} q^{i+j+1}.$$

Rewrite $IC_{i+j,i+j+1}$,

$$\begin{aligned} t^{i+j} - \kappa^{(i+j)} q^{i+j} &\geq (>) t^{i+j+1} - \kappa^{(i+j)} q^{i+j+1} \\ t^{i+j} - t^{i+j+1} &\geq (>) \kappa^{(i+j)} (q^{i+j} - q^{i+j+1}). \end{aligned}$$

Since $q^{i+j+1} \geq (>) q^{i+j}$ and $\kappa^{(i)} > \kappa^{(i+j)}$

$$\begin{aligned} t^{i+j} - t^{i+j+1} &\geq (>) \kappa^{(i)} (q^{i+j} - q^{i+j+1}) \\ t^{i+j} - \kappa^{(i)} q^{i+j} &\geq (>) t^{i+j+1} - \kappa^{(i)} q^{i+j+1}. \end{aligned}$$

This concludes the proof. $\square$

Next, we want to show that the relevant constraints “bind.” With non-integer costs but integer transfers, however, binding looks different here. Throughout the analysis, we will make

use of the ceiling function $\lceil \cdot \rceil$; see Appendix A for properties of the ceiling function. Notice that, for every $\kappa^{(i)}$ with $i = 1, \dots, m$, we have that $\lceil \kappa^{(i)} \rceil = \kappa^{(i)} + \frac{1}{\gamma}$.

By $IC_{i,i-1}$, we have:

$$\begin{aligned} t^i - \kappa^{(i)} q^i &\geq t^{i-1} - \kappa^{(i)} q^{i-1} \\ t^i &\geq t^{i-1} + \kappa^{(i)} (q^i - q^{i-1}). \end{aligned}$$

Now, $\kappa^{(i)}(q^i - q^{i-1})$ is not an integer. Thus, the smallest transfer t^i that observes $IC_{i,i-1}$ is:

$$\begin{aligned} t^i &= \left\lceil t^{i-1} + \kappa^{(i)} (q^i - q^{i-1}) \right\rceil \\ t^i &= t^{i-1} + \lceil \kappa^{(i)} (q^i - q^{i-1}) \rceil \quad \text{by Lemma A9 (vi).} \end{aligned} \tag{2}$$

In a standard setting with integer types, we would be able to write $t^i = \kappa^{(i)} q^i + t^{i-1} - \kappa^{(i)} q^{i-1}$ and obtain a characterization of transfers in terms of “cost plus information rents.” Unfortunately, it is not true in general that $\lceil \alpha n \rceil = \lceil \alpha \rceil n$ for an integer n ; for instance, $\lceil 2.5 \times 2 \rceil = 5 < 6 = \lceil 2.5 \rceil \times 2$. This is where our specification of types comes in handy.

Lemma 4 *For any $n \in D$ and $j = 1, \dots, m$, $\left\lceil \left(j - \frac{1}{\gamma}\right) n \right\rceil = \left\lceil j - \frac{1}{\gamma} \right\rceil n = jn$.*

PROOF. As $\gamma > b$ implies that $\frac{n}{\gamma} \in (0, 1)$ for any $n \in D$, we have $\left\lceil \left(j - \frac{1}{\gamma}\right) n \right\rceil = \left\lceil jn - \frac{n}{\gamma} \right\rceil = jn + \left\lceil -\frac{n}{\gamma} \right\rceil = jn$. $\square$

Lemma 5 *For all $i = 2, \dots, m$, $IC_{i,i-1}$ binds in the following sense:*

$$t^i = t^{i-1} + \left\lceil \kappa^{(i)} \right\rceil (q^i - q^{i-1}) \tag{3}$$

in the principal's optimum.

PROOF. Most of the argument is laid out in the text. From (2), apply Lemma 4. $\square$

Remark 1 *PC_1 is binding: $t^1 = \lceil \kappa^{(1)} \rceil q^1$.*

Combining Lemma 5 and Remark 1, we can write the transfers as follows:

$$t^i = \left\lceil \kappa^{(i)} \right\rceil q^i + \sum_{j=1}^{i-1} q^j.$$

As costs are non-integer but transfers must be integer, the latter involve a *round-up rent* that all types enjoy: $(\lceil \kappa^{(i)} \rceil - \kappa^{(i)}) q^i$ (aside from potential information rents). This round-up rent breaks the indifference and provides strict incentives for reporting truthfully and for participating.

Finally, we can replicate the result that local “downward” IC constraints (i.e., overstating your cost) ensure all local IC constraints.

Lemma 6 For all $i = 1, \dots, m-1$, $q_{i+1} \geq q_i$ ($q^{i+1} > q^i$) and $IC_{i,i-1}$ implies $IC_{i,i+1}$ (with strict inequality), i.e.,

$$t^i - \kappa^{(i)} q^i \geq (>) t^{i+1} - \kappa^{(i)} q^{i+1}.$$

PROOF. For any $i = 1, \dots, m-1$, we have that $t^{i+1} - t^i = \lceil \kappa^{(i+1)} \rceil (q^{i+1} - q^i)$. Since $\kappa^{(i)} = \kappa^{(i+1)} + 1$,

$$\begin{aligned} t^{i+1} - t^i &= \left\lceil \kappa^{(i)} - 1 \right\rceil (q^{i+1} - q^i) \\ t^{i+1} - t^i &= \left(\left\lceil \kappa^{(i)} \right\rceil - 1 \right) (q^{i+1} - q^i) \\ &= \left\lceil \kappa^{(i)} \right\rceil (q^{i+1} - q^i) - (q^{i+1} - q^i). \end{aligned}$$

Note that $-(q^{i+1} - q^i) < 0$ by monotonicity. Thus, $t^{i+1} - t^i < \lceil \kappa^{(i)} \rceil (q^{i+1} - q^i)$. Assume now, to the contrary, that $t^i - \kappa^{(i)} q^i \leq t^{i+1} - \kappa^{(i)} q^{i+1}$. Then,

$$\begin{aligned} t^i - \kappa^{(i)} q^i &\leq t^{i+1} - \kappa^{(i)} q^{i+1} \\ \kappa^{(i)} (q^{i+1} - q^i) &\leq t^{i+1} - t^i \\ \kappa^{(i)} (q^{i+1} - q^i) &\leq \left\lceil \kappa^{(i+1)} \right\rceil (q^{i+1} - q^i). \end{aligned}$$

With $q^{i+1} > q^i$, the inequality above leads to $\kappa^{(i)} < \lceil \kappa^{(i+1)} \rceil$, which is false:

$$\kappa^{(i+1)} + 1 = \kappa^{(i)} < \left\lceil \kappa^{(i+1)} \right\rceil \leq \kappa^{(i+1)} + 1.$$

Thus, the lemma follows. $\square$

Now, we can reduce the principal's problem as follows:

$$\max_{(q^i, t^i)_{i=1, \dots, m} \in (D^2)^m} \sum_{i=1}^m p^i (v(q^i) - t^i)$$

subject to, for all $i = 2, \dots, m$,

$IC_{i,i-1}$ for $i = 2, \dots, m$:

$$t^i = \left\lceil \kappa^{(i)} \right\rceil q^i + \sum_{j=1}^{i-1} q^j,$$

PC_1 :

$$t^1 = \left\lceil \kappa^{(1)} \right\rceil q^1,$$

$M_{i,i-1}$ for $i = 2, \dots, m$:

$$q^i \geq q^{i-1}.$$

We tentatively omit $M_{i,i-1}$, the monotonicity constraints. Substituting for the transfers in the objective function yields:

$$f(q^1, \dots, q^m) := \sum_{i=1}^m p^i \left(v(q^i) - \left\lceil \kappa^{(i)} \right\rceil q^i - \sum_{j=1}^{i-1} q^j \right) = \sum_{i=1}^m p^i \left(v(q^i) - \left(\left\lceil \kappa^{(i)} \right\rceil + \frac{\sum_{j>i} p^j}{p^i} \right) q^i \right),$$

where $\sum_{j>m} p^j \equiv 0$. The last equation follows from:

$$\sum_{i=1}^m p^i \sum_{j=1}^{i-1} q^j = \sum_{i=1}^m \left(\sum_{j>i}^m p^j q^i \right) = \sum_{i=1}^m p^i \left(\frac{\sum_{j>i}^m p^j}{p^i} q^i \right).$$

We are interested in maximizing $f(q^1, \dots, q^m)$, which we can do term by term. Thus, for each $i = 1, \dots, m$, we want to maximize:

$$v(q^i) - \left(\left\lceil \kappa^{(i)} \right\rceil + \frac{\sum_{j>i} p^j}{p^i} \right) q^i.$$

Employing the tools of discrete concave optimization presented in Appendix A, we derive the discrete first-order conditions for $i = 1, \dots, m$:

$$\begin{aligned} \Delta^+ v(q^i) - \left(\left\lceil \kappa^{(i)} \right\rceil + \frac{\sum_{j>i} p^j}{p^i} \right) &\leq 0 \\ \Delta^- v(q^i) - \left(\left\lceil \kappa^{(i)} \right\rceil + \frac{\sum_{j>i} p^j}{p^i} \right) &\geq 0, \end{aligned}$$

which we can combine as follows:

$$\Delta^+ v(q^i) \leq \left\lceil \kappa^{(i)} \right\rceil + \frac{\sum_{j>i} p^j}{p^i} \leq \Delta^- v(q^i). \quad (4)$$

The combined F.O.C.s are analogous to the usual calculus-based F.O.C.s.

Denote by φ^i the discrete version of the *virtual cost* of type $\kappa^{(i)}$:

$$\varphi^i := \left\lceil \kappa^{(i)} \right\rceil + \frac{\sum_{j>i}^m p^j}{p^i}.$$

We can write the F.O.C.s as $\Delta^+ v(q^i) \leq \varphi^i \leq \Delta^- v(q^i)$; see Figure 1. (The left panel of Figure 1 depicts schematically the principal's utility function with the optimum; the right panel shows how the discrete forward and backward derivatives of the value function “sandwich” the optimum at the corresponding virtual cost.) Notice that the lowest marginal cost type in Θ , $\kappa^{(m)}$, is commissioned the efficient amount of output (“no distortion at the top”), i.e.,

$$\Delta^+ v(q^m) \leq \left\lceil \kappa^{(m)} \right\rceil \leq \Delta^- v(q^m).$$

As in standard contracting, all other cost types suffer a downward distortion in the amount commissioned, in order to save on information rents.

The example below illustrates the computation of optimal quantities for the principal from the conditions in (4).

Example 1 Let $D = \{0, 1, \dots, 100\}$, $\Theta = \{0.99, 1.99, 2.99\}$ (here, $\gamma = 0.01$), and $v(q) = 50q - 0.5q^2$; we have $\Delta^+ v(q) = 49.5 - q$ and $\Delta^- v(q) = 50.5 - q$. We have $\lceil \kappa^{(1)} \rceil = 3$, $\lceil \kappa^{(2)} \rceil = 2$,

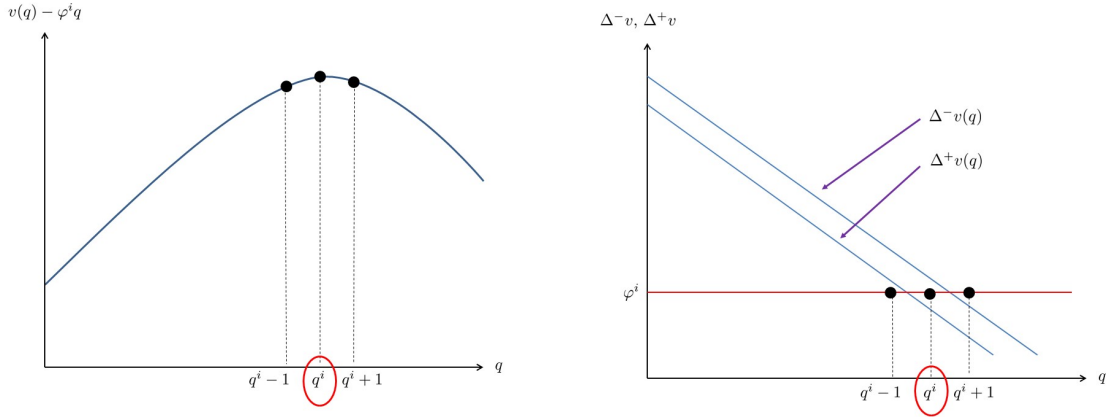


Figure 1: F.O.C.s for the reduced principal's problem for type $\kappa^{(i)}$.

and $\lceil \kappa^{(3)} \rceil = 1$. Consider the beliefs given by $p^1 = p^2 = p^3 = \frac{1}{3}$, which yield $\varphi^3 = 1$, $\varphi^2 = 3$, and $\varphi^1 = 5$. The F.O.C. for q^3 is:

$$49.5 - q^3 \leq 1 \leq 50.5 - q^3;$$

rearranging terms leads to $48.5 \leq q^3 \leq 49.5$, which yields $q^3 = 49$. For q^2 ,

$$49.5 - q^2 \leq 2 + \frac{p^3}{p^2} \leq 50.5 - q^2$$

$$49.5 - q^2 \leq 3 \leq 50.25 - 0.5q^2.$$

Now, we get $46.5 \leq q^2 \leq 47.5$, so $q^2 = 47$. Finally, for q^1 ,

$$49.5 - q^1 \leq 3 + \frac{p_2 + p_3}{p_1} \leq 50.5 - q^1$$

$$49.5 - q^1 \leq 5 \leq 50.5 - q^1;$$

we can rewrite the inequalities as $44.5 \leq q^1 \leq 45.5$, so we get $q^1 = 45$.

One notable difference with the case of continuum contracts is that, *even under strict concavity of $v(q)$* , the F.O.C.s need not have a unique solution. We illustrate non-uniqueness in the following example.

Example 2 Consider the same setting as in example 1, but now let the beliefs be given by $p^1 = p^3 = 0.25$, and $p^2 = 0.5$. The F.O.C.s for q^3 is independent of the beliefs, so we once again get $q^3 = 49$. For q^1 , we now have:

$$49.5 - q^1 \leq 3 + \frac{p_2 + p_3}{p_1} \leq 50.5 - q^1$$

$$49.5 - q^1 \leq 6 \leq 50.5 - q^1;$$

we can rewrite the inequalities as $43.5 \leq q^1 \leq 44.5$, so we get $q^1 = 44$. Finally, for q^2 ,

$$49.5 - q^2 \leq 2 + \frac{p_3}{p_2} \leq 50.5 - q^2$$

$$49.5 - q^2 \leq 2.5 \leq 50.5 - q^2.$$

Now, we get $47 \leq q^2 \leq 48$, so we have two possible solutions: $q^2 = 47$ and $q^2 = 48$.

In fact, for any type, there can be two solutions—but no more than two, and they must be adjacent. For any type $\kappa^{(i)}$, q^i and $q^i + 1$ are the two solutions if and only if $\Delta^-v(q^i + 1) = \Delta^+v(q^i) = \varphi^i$. Lemma A8 in the appendix provides a general proof, but we present the statement in the context of our principal's problem below.

Lemma 7 *For each $i = 1, \dots, m$, there are at most two global maximizers. If q^i and $q^{i'}$ are the global maximizers, then either $q^i = q^{i'}$ or $\max\{q^i, q^{i'}\} = \min\{q^i, q^{i'}\} + 1$. Moreover, if $q^i > q^{i'}$, then q^i and $q^{i'}$ are both global maximizers if and only if $\Delta^-v(q^i) = \varphi^i = \Delta^+v(q^{i'})$.*

Figure 2 illustrates the condition identified in Lemma 7. The two optimal quantities attain the exact same value of $v(q) - \varphi^i q$. If quantities could be chosen from a continuum and $v(q)$ were extended accordingly, strict concavity implies that all intermediate values for q are even better; however, such intermediate values are infeasible under integer contracts.

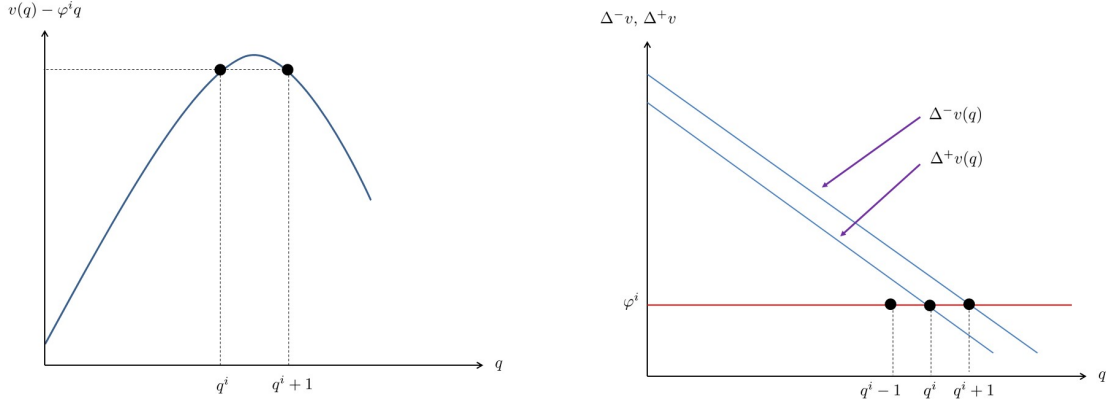


Figure 2: There are two different optimal quantities for type $\kappa^{(i)}$.

We now turn to the ignored monotonicity constraints. With finite types, a commonly-assumed sufficient condition for monotonicity not to bind is log-concavity of p .

Assumption 2 *Probability distribution $p \in \Delta(\Theta)$ is log-concave if for all $i = 2, \dots, m - 1$, $p^i p^i \geq p^{i+1} p^{i-1}$.*

We denote the set of full-support, log-concave distributions as $\Delta^{FLC}(\Theta) \subset \text{int}(\Delta(\Theta))$. Lemma B1 in the appendix shows that Assumption 2 leads to a non-increasing likelihood ratio:

$$\frac{p^{i+k}}{p^i} \geq \frac{p^{j+k}}{p^j}$$

for any $i, j = 1, \dots, m$ and k such that $j > i$ and $j + k \leq m$; and to a non-increasing discrete inverse hazard rate or *Mills' ratio*: For any $i, j = 1, \dots, m$ with $j > i$, $m_d^i \geq m_d^j$, where:

$$m_d^i := \frac{\sum_{k>i}^m p^k}{p^i}.$$

It can also be shown that Assumption 2 yields decreasing virtual costs: $\varphi^{i+1} < \varphi^i$ for all $i = 1, \dots, m-1$ (see Lemma B2 in the appendix), and implies monotonicity of the allocations.

Proposition 1 *Under Assumptions 1 and 2, the F.O.C.s imply the monotonicity constraints $M_{i,i-1}$ for all $i = 1, \dots, n-1$. If for some $i, i+1$ we have two distinct solutions $q^i, \hat{q}^i$ and $q^{i+1}, \hat{q}^{i+1}$, then $\min\{q^{i+1}, \hat{q}^{i+1}\} \geq \max\{q^i, \hat{q}^i\}$.*

PROOF. For $i = 1, \dots, m-1$, let q^i and q^{i+1} be solutions to the corresponding F.O.C.s, $\Delta^+v(q^i) \leq \varphi^i \leq \Delta^-v(q^i)$ and $\Delta^+v(q^{i+1}) \leq \varphi^{i+1} \leq \Delta^-v(q^{i+1})$. Assume that we have some $i = 1, \dots, m-1$ such that $q^i > q^{i+1}$. Then, by Lemma A6 (iv) in the appendix, $\Delta^-v(q^i) \leq \Delta^+v(q^{i+1})$. Combining the conditions, we get:

$$\varphi^i \leq \Delta^-v(q^i) \leq \Delta^+v(q^{i+1}) \leq \varphi^{i+1},$$

which violates Assumption 2. □

While Assumption 2 is sufficient for monotonicity of the quantities, it does not guarantee strict monotonicity *even if the discrete Mills' ratio is strictly monotone*. This is demonstrated in the next example.

Example 3 *Consider the utility function $v(q) = 253q - 2.5q^2$, types $\Theta = \{0.99, 1.99, 2.99\}$, and beliefs $p^1 = p^2 = p^3 = \frac{1}{3}$. Here, $\Delta^+v(q) = 250.5 - 5q$, $\Delta^-v(q) = 255.5 - 5q$, $\varphi^1 = 5$, $\varphi^2 = 3$, and $\varphi^3 = 1$. We have:*

$$\begin{aligned} 250.5 - 5q^3 &\leq 1 \leq 255.5 - 5q^3 \Rightarrow q^3 = 50; \\ 250.5 - 5q^2 &\leq 3 \leq 255.5 - 5q^2 \Rightarrow q^2 = 50; \\ 250.5 - 5q^1 &\leq 5 \leq 255.5 - 5q^1 \Rightarrow q^1 = 50. \end{aligned}$$

The issue in Example 3 is that $v(q)$ is “too concave”: The absolute value of the slope of $\Delta^+v(q)$ and $\Delta^-v(q)$ is (weakly) greater than all virtual utilities $\varphi^1, \varphi^2, \varphi^3$. Thus, when dividing to solve for the quantities, the endpoints of the quantity ranges end up differing by less than 1 and “trap” the same integer; see Figure 3.

In some applications (especially when the screening problem is to be solved using rationalizability as we demonstrate in the next section), we may want monotonicity to hold strictly, so as to provide strict incentives for the agent. To ensure strict monotonicity, we can restrict the “degree of concavity” of $v(q)$ by setting a lower-bound on $\Delta^+\Delta^-v(q)$.³

Assumption 3 *For every $q \in D$, $\Delta^+\Delta^-v(q) \geq -1$.*

³Recall that Assumption 1.3 can be written as $0 > \Delta^+\Delta^-v(q)$.

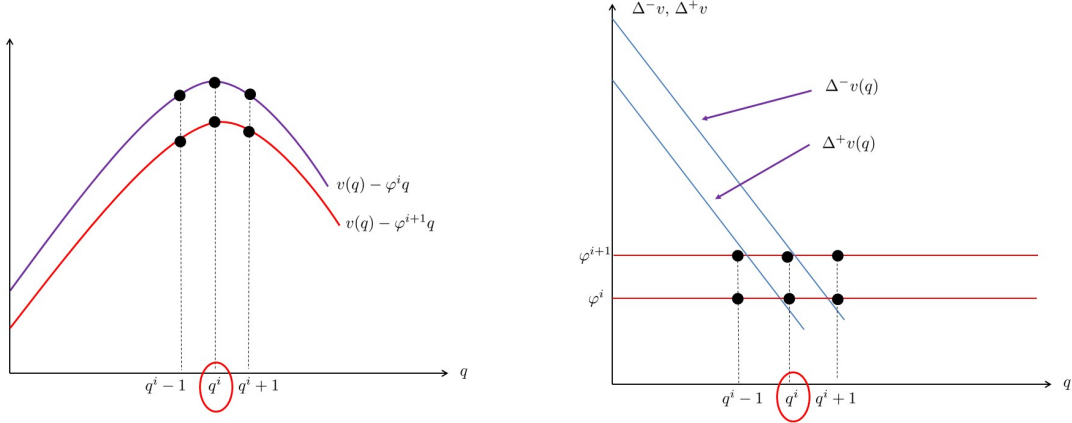


Figure 3: Types $\kappa^{(i)}$ and $\kappa^{(i+1)}$ are awarded the same quantity.

Proposition 2 *Under Assumptions 1 through 3, the F.O.C.s imply $M_{i,i-1}$ for all $i = 1, \dots, m-1$ with strict inequality; i.e., $q^{i+1} > q^i$. If for some $i, i+1$ we have two distinct solutions $q^i, \hat{q}^i$ and $q^{i+1}, \hat{q}^{i+1}$, then $\min\{q^{i+1}, \hat{q}^{i+1}\} > \max\{q^i, \hat{q}^i\}$.*

PROOF. Assumption 2 leads to weak monotonicity, so it remains to show that $q^{i+1} \neq q^i$. By Assumption 3,

$$\begin{aligned}
 \Delta^+ \Delta^- v(q^i) &\geq -1 \\
 \Delta^+ \Delta^- v(q^i) &\geq \frac{\sum_{j>i+1}^m p^j}{p^{i+1}} - \frac{\sum_{j>i}^m p^j}{p^i} - 1 \\
 \Delta^- v(q^i + 1) - \Delta^- v(q^i) &\geq \frac{\sum_{j>i+1}^m p^j}{p^{i+1}} - \frac{\sum_{j>i}^m p^j}{p^i} + \kappa^{(i+1)} - \kappa^{(i)} \\
 \Delta^+ v(q^i) - \Delta^- v(q^i) &\geq \varphi^{i+1} - \varphi^i.
 \end{aligned} \tag{5}$$

From Inequalities (4), it follows that $\Delta^+ v(q^{i+1}) - \Delta^- v(q^i) < \varphi^{i+1} - \varphi^i$.

Now, suppose that $q^{i+1} = q^i$. Then, we have $\Delta^+ v(q^i) - \Delta^- v(q^i) < \varphi^{i+1} - \varphi^i$, a contradiction to Inequality (5).

Finally, suppose that the F.O.C.s for type $\kappa^{(i)}$ has two solutions. Then, $\Delta^- v(q^i + 1) = \Delta^+ v(q^i) = \varphi^i$. By the previous argument, $q^{i+1} \geq q^i + 1$. Assume, to the contrary, that $q^{i+1} = q^i + 1$. Then,

$$\begin{aligned}
 \Delta^+ v(q^{i+1}) &\leq \varphi^{i+1} \leq \Delta^- v(q^{i+1}) \\
 \Delta^+ v(q^i + 1) &\leq \varphi^{i+1} \leq \Delta^- v(q^i + 1) \\
 \Delta^+ v(q^i + 1) &\leq \varphi^{i+1} \leq \Delta^- v(q^i + 1) = \varphi^i.
 \end{aligned}$$

Rewriting the last line and taking advantage of Lemma B2 in the appendix, we obtain

$$\Delta^+ v(q^i + 1) \leq \varphi^{i+1} < \varphi^i = \Delta^- v(q^i + 1),$$

which violates Inequality (5) for q^{i+1} . □

4 Optimal Menus

For a given belief $p \in \Delta^{FLC}(\Theta)$ and type $\kappa^{(i)}$, we denote by $Q^i(p)$ the set of optimal quantities. Under Assumptions 1 through 3, we have:

$$Q^i(p) = \{q \in D : \Delta^+ v(q) \leq \varphi^i(p) \leq \Delta^- v(q)\},$$

where we now write $\varphi^i(p)$ to make explicit the dependence of virtual costs on beliefs. Recall that sets $Q^i(p)$ are at least singletons and at most doubletons consisting of quantities that are adjacent to each other on the discrete concave screening D .

We define a menu of optimal contracts as a menu consisting of exactly m contracts, with exactly one of them for each marginal cost-type, that together solve the F.O.C.s in (4) given p . The set of menus of optimal contracts is:⁴

$$\mathcal{M}^*(p) := \left\{ M \in \mathcal{M} : \begin{array}{l} |M| = m \text{ and } \forall i = 1, \dots, m \exists!(q^i, t^i) \in M \\ (q^i \in Q^i(p), t^i = \lceil \kappa^{(i)} \rceil q^i + \sum_{j=1}^{i-1} q^j) \end{array} \right\}.$$

When the solution to the principal's problem with belief p is unique, then $\mathcal{M}^*(p)$ is a singleton, i.e., it consists of just one menu. When the optimal contract for a given type is non-unique, the principal could offer both. In this setting, we show that the agent strictly prefers the contract with the higher of the two quantities, as it means earning a higher round-up rent.

Lemma 8 *Given p , let both $M, \tilde{M} \in \mathcal{M}^*(p)$ be two different optimal menus for the principal for which there exists some type of the agent $\kappa^{(i)}$ such that $(q^i, t^i) \in M$, $(\tilde{q}^i, \tilde{t}^i) \in \tilde{M}$, and $q^i = \tilde{q}^i + 1$. Then, $u_A(\kappa^{(i)}, (q^i, t^i)) > u_A(\kappa^{(i)}, (\tilde{q}^i, \tilde{t}^i))$.*

PROOF. We prove by induction on the marginal cost types.

Base case $i = 1$: For the optimal (q^1, t^1) , we have:

$$\begin{aligned} u_A(\kappa^{(1)}, q^1, t^1) &= \lceil \kappa^{(1)} \rceil q^1 - \kappa^{(1)} q^1 \\ &= \left\lceil m - \frac{1}{\gamma} \right\rceil q^1 - \left(m - \frac{1}{\gamma} \right) q^1 \\ &= m q^1 - \left(m - \frac{1}{\gamma} \right) q^1 = \frac{1}{\gamma} q^1, \end{aligned}$$

which is strictly increasing in q^1 . Thus, we have $u_A(\kappa^{(1)}, (q^1, t^1)) \geq u_A(\kappa^{(1)}, (\tilde{q}^1, \tilde{t}^1))$, with strict inequality if $q^1 \neq \tilde{q}^1$.

Induction hypothesis: For $i = 2, \dots, m-1$, $u_A(\kappa^{(i)}, (q^i, t^i)) \geq u_A(\kappa^{(i)}, (\tilde{q}^i, \tilde{t}^i))$, with strict inequality if $q^j \neq \tilde{q}^j$ for some $j = 1, \dots, i$.

Inductive step $i = m$: Observe that:

$$u_A(\kappa^{(i)}, q^i, t^i) = \lceil \kappa^{(i)} \rceil q^i + \sum_{j=1}^{i-1} q^j - \kappa^{(i)} q^i = \frac{1}{\gamma} q^i + \sum_{j=1}^{i-1} q^j,$$

⁴As customary, we let “ $\exists!$ ” stand for “There exists a unique”.

which is increasing in q^j for $j = 1, \dots, i-1$. Thus, $u_A(\kappa^{(i)}, (q^i, t^i)) \geq u_A(\kappa^{(i)}, (\tilde{q}^i, \tilde{t}^i))$ with strict inequality if q^j is different for some j from $\tilde{q}^j$. $\square$

Non-uniqueness of optimal contacts for a given type is not ruled out by strict concavity, strict incentives, or strict monotonicity. Nonetheless, it is—mostly—a knife-edge case with respect to beliefs. If q^i and $q^{i'} < q^i$ are two different solutions for type $\kappa^{(i)}$, then $\Delta^-v(q^i) = \varphi^i(p) = \Delta^+v(q^{i'})$ (Lemma 7, Figure 2) and a small perturbation of the beliefs can yield uniqueness. The exception is non-monotonicity of the contract designed for the lowest-cost agent, as the corresponding F.O.C.s do not involve the beliefs. To tackle this last case, we introduce one last assumption on $v(q)$.

Assumption 4 *For every $q \in \{0, \dots, b-1\}$ and $n \in \mathbb{N}$, $\Delta^+v(q) \neq n$.*

Assumption 4 ensures uniqueness of q^m by ruling out the case $\Delta^+v(q^m) = \lceil \kappa^{(m)} \rceil$. For the other types, as noted, we can simply perturb beliefs.

Lemma 9 *Under Assumptions 1 through 4, there exists a nonempty open set of full support beliefs $O \subseteq \Delta^{FLC}(\Theta)$ such that $\mathcal{M}^*(p) = \mathcal{M}^*(p')$ for all $p, p' \in O$ if and only if $\mathcal{M}^*(p) = \{M\}$.*

PROOF. “ $\Leftarrow$ ”: If there is a full support belief p such that $\mathcal{M}^*(p) = \{M\}$, then by Lemma 7, we have for all types $\kappa^{(i)}$ with $i = 1, \dots, m-1$, $\Delta^+v(q^i) < \varphi^i(p)$. Then, there exists an open set O around p such that for any $\tilde{p} \in O$ we still have $\Delta^+v(q^i) < \varphi^i(\tilde{p})$.

“ $\Rightarrow$ ”: Suppose by contradiction that there exists a nonempty open ball of full support beliefs $O \subseteq \Delta^{FLC}(\Theta)$ such that $\mathcal{M}^*(p) = \mathcal{M}^*(p')$ for all $p, p' \in O$ but $|\mathcal{M}^*(p)| > 1$. Non-uniqueness of the optimal menu of contracts implies, by Lemma 7, that there is a type $\kappa^{(i)}$ with $i = 1, \dots, m-1$ for which $\Delta^+v(q^i) = \varphi^i(p)$. Since O is open, there exists $\tilde{p} \in O$ such that $\Delta^+v(q^i) < \varphi^i(\tilde{p})$, implying that $\mathcal{M}^*(\tilde{p})$ is a singleton, a contradiction. $\square$

If adding contracts to a menu is costless, there is still another sense in which we have non-uniqueness of optimal menus: adding “irrelevant” contracts. Any menu M that is a strict superset of a menu of optimal contracts is also optimal provided that the additional contracts are never selected by the agent.

Thus, we can expand $\mathcal{M}^*(p)$ as follows. For any full-support p such that $|\mathcal{M}^*(p)| \neq 1$, let $\mathcal{M}(p) = \emptyset$. Otherwise, let:

$$\mathcal{M}(p) := \left\{ M \in \mathcal{M} : \begin{array}{l} \mathcal{M}^*(p) = \{M'\}, M' \subseteq M \text{ and } \forall \mathbf{c} \in M \setminus M' \\ (\nexists \theta \in \Theta (\mathbf{c} \in \arg \max_{\mathbf{c}' \in M} u_A(\mathbf{c}', \theta))) \end{array} \right\}.$$

That is, if $\mathcal{M}(p)$ is non-empty, then any menu in $\mathcal{M}(p)$ includes the unique menu of optimal contracts given p and additional contracts which no type of the agent chooses.

If we are interested in standard, equilibrium screening with integer types, then $\mathcal{M}^*(p)$ is the set of optimal or equilibrium menus for the common prior $p \in \Delta^{FLC}(\Theta)$. Accommodating indifferences in incentive and participation constraints as usual, we can drop Assumptions (3) and (4).

5 Rationalizable Screening

Francetich and Schipper (2025) employs the present setup in a problem where the principal is initially unaware of some of the agent's types, and the agent may raise the principal's awareness of some such types. As is typical in games with unawareness, we employ a rationalizability approach because equilibrium lacks foundation when surprises can happen on equilibrium path (Heifetz et al., 2013). This begs the question about rationalizable outcomes in screening without unawareness. In this section, we provide the benchmark of rationalizability outcomes in screening under full awareness. The finiteness of our setting ensures that the best responses are non-empty valued and that higher-order beliefs are well defined. The presence of round-up rents allows us to avert ties in incentive compatibility and participation constraints, which cannot be resolved with an equilibrium tie-breaking assumption under rationalizability. Finally, the robustness component of our rationalizability notion (along with Assumption 4) ensures uniqueness of optimal quantities. We show that our rationalizability notion selects the (augmented) optimal menu from Section 4. In a sense, we show that equilibrium is not necessary to solve the screening problem.

The game proceeds in three stages: In the first stage, nature selects the payoff type of the agent, $\theta \in \Theta$. In the second stage, the principal chooses a menu of contracts $M \in \mathcal{M}$ to be offered to the agent. In the third and final stage, the agent selects a contract $\mathbf{c} = (q, t) \in M$ or his outside option $\mathbf{o}$. Since the payoff type of the agent is his private information, the principal's strategy space is simply $\mathcal{M}$. The agent's information sets can be identified with tuples $(\theta, M) \in \Theta \times \mathcal{M}$. His strategies are maps $s : \Theta \times \mathcal{M} \rightarrow \{(q, t) : q, t \in D\}$ with $s(\theta, M) \in M \cup \{\mathbf{o}\}$. Denote by $S \subseteq (D^2)^{\Theta \times \mathcal{M}}$ the agent's set of strategies.

The principal forms beliefs about marginal cost-types and strategies of the agent, $\beta_P \in \Delta(\Theta \times S)$. We let Δ be the subset of $\Delta(\Theta \times S)$ such that the marginal beliefs on Θ are full support and log-concave:

$$\Delta = \{\beta_P \in \Delta(\Theta \times S) : \text{marg}_{\Theta} \beta_P \in \Delta^{FLC}(\Theta)\}.$$

Since the agent has perfect information at each information set, his beliefs are degenerate.

Given his type, the agent chooses either a contract from the given menu or the outside option. Thus, his best-response correspondence is:

$$BR_A(\theta, M) := \arg \max_{\mathbf{c} \in M \cup \{\mathbf{o}\}} u_A(\mathbf{c}, \theta).$$

Next, let $U_P(M, \beta_P)$ denote the principal's expected payoff from offering contract-menu M given her belief $\beta_P \in \Delta$:

$$U_P(M, \beta_P) := \sum_{(\theta, s) \in \Theta \times S} \beta_P(\{(\theta, s)\}) \cdot u_P(s(\theta, M)).$$

Her best response correspondence is:

$$BR_P(\beta_P) := \arg \max_{M \in \mathcal{M}} U_P(M, \beta_P).$$

Our rationalizability notion, which we call Δ -O rationalizability, combines the idea of Δ -rationalizability with best replies being robust to small perturbations of beliefs. Formally:

Definition 1 (Δ -O Rationalizability) Define recursively $R_P^0 = \mathcal{M}$ and $R_A^0 = S$ and, for $k \geq 1$,

$$\begin{aligned} B_P^k &= \left\{ \beta_P \in \Delta : \text{marg}_S \beta_P \left(R_A^{k-1} \right) = 1 \right\}; \\ R_P^k &= \left\{ M \in R_P^{k-1} : \exists \text{ non-empty open set } O \subseteq B_P^k \left(M \in \bigcap_{\beta_P \in O} BR_P(\beta_P) \right) \right\}; \\ R_A^k &= \left\{ s \in R_A^{k-1} : \forall (\theta, M) \in \Theta \times \mathcal{M} (s(\theta, M) \in BR_A(\theta, M)) \right\}. \end{aligned}$$

For $i \in \{P, A\}$, the set of player i 's Δ -O rationalizable strategies is $R_i^\infty = \bigcap_{k=1}^\infty R_i^k$.

At each level k of the procedure, the principal forms beliefs about the agent's types and strategies such that: (a) Her marginal belief over the agent's types has full support and is log-concave (by the requirement that beliefs are in Δ), and (b) she is certain of the $(k-1)$ -level Δ -O rationalizable strategies of the agent. She then selects a $(k-1)$ -level rationalizable strategy for which there exists an open set of k -level beliefs such that the strategy is a best response to *any* belief in this set. The agent at level k selects a $(k-1)$ -level Δ -O rationalizable strategy such that for any cost-type of his and menu of contracts received from the principal, he selects his best contract as long as his participation constraint is satisfied, and his outside option otherwise.

Theorem 1 Under Assumptions 1 through 4, $s_P^\infty \in R_P^\infty$ if and only if there exists some $p \in \Delta^{FLC}(\Theta)$ such that $s_P^\infty \in \mathcal{M}(p)$. In other words, the set of Δ -O rationalizable menus of contracts is $\bigcup_{p \in \Delta^{FLC}(\Theta)} \mathcal{M}(p)$.

PROOF. The proof is by induction on levels of elimination.

Level 1, principal. Define $\mathcal{M}^* := \{M \in \mathcal{M} : \exists (q, t) \in M (v(q) - t > 0)\}$.

We claim that $R_P^1 = \mathcal{M}^*$. First, we show (the contrapositive of) $R_P^1 \subseteq \mathcal{M}^*$. Suppose to the contrary that there is a menu $M \in R_P^1$ such that for all $(q, t) \in M$, $v(q) - t \leq 0$; i.e., $M \notin \mathcal{M}^*$. Such a menu yields a non-positive expected payoff to the principal for any $\beta_P \in B_P^1$. Compare this to a menu M' for which there exists $(q, t) \in M'$ such that $v(q) - t > 0$ and for all $(q, t) \in M'$, $v(q) - t \geq 0$. Such a menu M' yields a strict positive expected payoff to the principal for any full support belief (which are contained in B_P^1), a contradiction to $M \in R_P^1$.

Next, we show $\mathcal{M}^* \subseteq R_P^1$. Consider a menu $M \in \mathcal{M}^*$. Let $\beta_P \in B_P^1$ assign sufficiently large probability to the agent taking contract $(q, t) \in M$ with $v(q) - t > 0$ when faced with menu M and $\mathbf{o}$ when faced with any other menu. By continuity of von-Neumann-Morgenstern expected utility, there exists an open ball O around β_P such that M is a best response to any $\beta'_P \in O$. Thus, $M \in R_P^1$.

Level 1, agent. For any menu offered by the principal, the agent selects the profit maximizing contract given his type yielding non-negative profits; if there is no contract in the menu yielding non-negative profits, the agent selects the outside option. Formally, $R_A^1 = \{s \in S : \forall (\theta, M) \in \Theta \times \mathcal{M} (s(\theta, M) \in BR_A(\theta, M))\}$.

Level 2, principal. The principal is now certain of R_A^1 . Thus, for any $\beta_P \in B_P^2$, she is certain that the agent observes incentive compatibility and participation constraints. We claim that $R_P^2 = \bigcup_{\beta_P \in B_P^2} \mathcal{M}(\text{marg}_\Theta \beta_P)$.

“ $\supseteq$ ”: There is nothing to prove if $\mathcal{M}(\text{marg}_\Theta \beta_P) = \emptyset$, so take $\beta_P \in B_P^2$ such that $\mathcal{M}(\text{marg}_\Theta \beta_P) \neq \emptyset$. By Lemma 9, there exists a nonempty open ball $O \subseteq B_P^2$ about β_P such that $M \in \bigcap_{\beta'_P \in O} BR_P(\beta'_P)$. Hence, $\mathcal{M}(\text{marg}_\Theta \beta_P) \subseteq R_P^2$.

“ $\subseteq$ ”: If $M \in R_P^2$, then there exists a nonempty open set $O \subseteq B_P^2$ such that $M \in \bigcap_{\beta'_P \in O} BR_P(\beta'_P)$. For any such $\beta_P \in O$, there exists $M^* \in \mathcal{M}^*(\text{marg}_\Theta \beta_P)$ with $M^* \subseteq M$. Note that the mapping from O to $\mathcal{M}$ defined by $\mathcal{M}^*(\text{marg}_\Theta \beta_P)$ cannot be bijective since O is infinite but $\mathcal{M}$ is finite. Then, there exists a nonempty open set $O' \subseteq O$ such that $\mathcal{M}^*(\text{marg}_\Theta \beta'_P) = \mathcal{M}^*(\text{marg}_\Theta \beta''_P) = M^*$ for all $\beta'_P, \beta''_P \in O'$, where the last equality follows from Lemma 9. By definition of $\mathcal{M}(\cdot)$, $M \in \mathcal{M}(\text{marg}_\Theta \beta'_P)$ for all $\beta'_P \in O'$.

Level 2, agent. No further change. No matter what menu the agent it presented with, he chooses a best response and thus $R_A^2 = R_A^1$.

Level 3, principal. No further change. Since $R_A^2 = R_A^1$, we must have $B_P^3 = B_P^2$. Thus, $R_P^3 = R_P^2$.

Level 3, agent. No further change. No matter what menu the agent is presented with, he chooses a best response and thus $R_A^3 = R_A^2$.

The maximal reduction of strategies is reached after two levels of the procedure; thus, we have: $R_A^\infty = R_A^2$, $R_P^\infty = R_P^2$. Every outcome coincides with an equilibrium outcome for some full-support, log-concave marginal belief on Θ —possibly with additional, irrelevant contracts. $\square$

We conclude that under Assumptions 1 through 4, for all $p \in \Delta^{FLC}(\Theta)$, the standard text-book equilibrium menu is in $\mathcal{M}^*(p)$ and Δ -O-rationalizable. Even if the incentive compatibility constraints bind for the standard text-book equilibrium menu, we can perturb slightly so that ICs hold strictly. Conversely, we have that for any Δ -O rationalizable menu, there is a common prior $p \in \Delta^{FLC}(\Theta)$ for which this menu is a standard text-book equilibrium menu.

6 Conclusion

We analyze a contract-design problem in a setting with finite, non-integer marginal cost types and finite, integer contracts. Our modeling of non-integer marginal costs allows us to replicate the typical constraint-simplification results and thus to emulate the well-treaded steps of screening under a continuum of contracts.

Employing discrete derivatives, we characterize the optimal contracts. We show that, in the discrete setting, the solutions to the discrete F.O.C.s need not be unique *even under discrete strict concavity*. Moreover, log-concavity of the distribution of types can only ensure weak monotonicity of the quantities *even if virtual costs are strictly monotone*.

Under our specification of non-integer costs, all types of the agent enjoy a round-up rent. Thus, the relevant incentive compatibility and participation constraints “bind” but with strict inequality. Breaking indifferences simplifies the rationalizability analysis and, along with our other assumptions, allows us to derive sharp predictions. We show that our rationalizability notion, Δ -O rationalizability, selects the set of usual optimal contracts—possibly along with irrelevant contracts.

Our rationalizability analysis serves as the full-awareness benchmark for Francetich and Schipper (2025), where the principal is initially unaware of some of the agent’s types. In said paper, we explore the agent’s incentive to raise the principal’s awareness—either fully or partially.

A Discrete Concave Optimization

We briefly review the basic elements of discrete concave optimization employed in the paper. Denote by $\mathbb{N} = \{1, 2, 3, \dots\}$ the set of natural numbers and by $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. A function $f : \mathbb{N}_0 \rightarrow \mathbb{R}$ is discrete concave if for all $x \in \mathbb{N}$,

$$f(x+1) + f(x-1) \leq 2f(x).$$

It is discrete strictly concave if the inequality holds strictly.

We say that x is a local maximizer of f if $f(x) \geq \max\{f(x+1), f(x-1)\}$. It is a strict local maximizer if the inequality holds strictly. We say that x is a global maximizer of f if $f(x) \geq f(y)$ for all $y \in \mathbb{N}_0$. It is a unique global maximizer if the inequality holds strictly for all $y \neq x$.

Lemma A1 *If f is discrete concave, then x is a global maximizer if and only if it is a local maximizer. Moreover, x is the unique global maximizer if and only if it is a strict local maximizer.*

PROOF. Let f be discrete concave. The “only if” direction is trivial. We prove the “if” direction. For any $n \in \mathbb{N}$,

$$0 \leq f(x) - f(x+1) \leq f(x+1) - f(x+2) \leq f(x+2) - f(x+3) \leq \dots \leq f(x+n-1) - f(x+n),$$

where the first inequality follows from x being a local maximizer of f and the other inequalities follow from discrete concavity. This implies $f(x) \geq f(x+1)$, $f(x+1) \geq f(x+2)$, $f(x+2) \geq f(x+3)$, ..., $f(x+n-1) \geq f(x+n)$. We conclude $f(x) \geq f(x+n)$ for any $n \in \mathbb{N}$.

Similarly, for any $n \in \mathbb{N}$,

$$0 \geq f(x-1) - f(x) \geq f(x-2) - f(x-1) \geq f(x-3) - f(x-2) \geq \dots \geq f(x-n+1) - f(x-n),$$

where the first inequality follows from x being a local maximizer of f and other inequalities follow from discrete concavity. This implies $f(x) \geq f(x-1)$, $f(x-1) \geq f(x-2)$, $f(x-2) \geq f(x-3)$, ..., $f(x-n+1) \geq f(x-n)$. We conclude $f(x) \geq f(x-n)$ for any $n \in \mathbb{N}$. Thus, x is a global maximizer.

The version of the argument for the unique global maximizer follows from strict local maximizer implying that the first inequality in each chain above is a strict inequality. $\square$

Define the discrete forward derivative of f at $x \in \mathbb{N}_0$ by

$$\Delta^+ f(x) := f(x+1) - f(x)$$

and the discrete backward derivative of f at $x \in \mathbb{N}$ by

$$\Delta^- f(x) := f(x) - f(x-1).$$

Clearly, for any $x \in \mathbb{N}_0$,

$$\Delta^+ f(x) = \Delta^- f(x+1)$$

and for any $x \in \mathbb{N}$,

$$\Delta^- f(x) = \Delta^+ f(x-1).$$

The following observation is immediate from the definition.

Lemma A2 *The following statements are equivalent:*

1. *Function f is monotone nondecreasing, i.e., $x \geq y$ implies $f(x) \geq f(y)$.*
2. *$\Delta^+ f(x) \geq 0$ for all $x \in \mathbb{N}_0$.*
3. *$\Delta^- f(x) \geq 0$ for all $x \in \mathbb{N}$.*

It is monotone strictly increasing if and only if the inequalities hold strictly.

The following observation on “first-order conditions” for optimization involving discrete derivatives is immediate from the definitions.

Lemma A3 (“First-order conditions”) *We have $x \in \mathbb{N}_0$ is a local maximizer of f if and only if*

$$\Delta^- f(x) \geq 0 \text{ and } \Delta^+ f(x) \leq 0$$

whenever these expressions are defined for x . It is a strict local maximizer if and only if the inequalities hold strictly whenever these expressions are defined for x .

Denote by $\Delta^+ \Delta^+ f(x)$ the second discrete forward derivative of f at x defined by $\Delta^+(\Delta^+ f(x))$ (and analogously for $\Delta^- \Delta^- f(x)$, $\Delta^- \Delta^+ f(x)$, and $\Delta^+ \Delta^- f(x)$).

Lemma A4 *For any $x \in \mathbb{N}$, $\Delta^+ \Delta^- f(x) = \Delta^- \Delta^+ f(x)$.*

PROOF. $\Delta^+ \Delta^- f(x) = \Delta^- f(x+1) - \Delta^- f(x) = f(x+1) - f(x) - f(x) + f(x-1) = \Delta^+ f(x) - \Delta^+ f(x-1) = \Delta^- \Delta^+ f(x)$. $\square$

The following observation is immediate from the definitions.

Lemma A5 *The following statements are equivalent:*

1. *The function f is discrete concave.*
2. *For all $x \in \mathbb{N}$, $\Delta^+ f(x) \leq \Delta^- f(x)$.*
3. *For all $x \in \mathbb{N}_0$, $\Delta^+ \Delta^+ f(x) \leq 0$.*

4. For all $x \in \mathbb{N}$, $\Delta^- \Delta^- f(x-1) \leq 0$.

5. For all $x \in \mathbb{N}$, $\Delta^+ \Delta^- f(x) \leq 0$.

6. For all $x \in \mathbb{N}$, $\Delta^- \Delta^+ f(x) \leq 0$.

The inequalities hold strictly if and only if f is discrete strictly concave.

Lemma A6 *If f is discrete concave, then $x > y$ implies:*

$$(i) \quad \Delta^- f(x) \leq \Delta^- f(y),$$

$$(ii) \quad \Delta^+ f(x) \leq \Delta^+ f(y),$$

$$(iii) \quad \Delta^+ f(x) \leq \Delta^- f(y),$$

$$(iv) \quad \Delta^- f(x) \leq \Delta^+ f(y).$$

PROOF. (i) W.l.o.g., let $y \in \mathbb{N}$ and take $x := y + n$ for some $n \in \mathbb{N}$. Observe that:

$$f(x) - f(x-1) \leq f(x-1) - f(x-2) \leq \dots \leq f(y) - f(y-1),$$

where each inequality follows from f being discrete concave. This chain of inequalities is equivalent to:

$$\Delta^- f(x) \leq \Delta^- f(x-1) \leq \dots \leq \Delta^- f(y).$$

We conclude (i).

Item (ii) follows analogously.

For (iii), apply (2.) from the previous lemma to either (i) or (ii).

For (iv), note that by the definitions of discrete derivatives, $\Delta^- f(x) = \Delta^+ f(x-1)$. The conclusion follows now from (ii). $\square$

Note that for $y = x-1$, property (iv) holds with equality irrespective of whether $f(x)$ is discrete concave.

Lemma A7 *Let f is discrete strictly concave.*

$$(i) \quad x > y \text{ if and only if } \Delta^- f(x) < \Delta^- f(y),$$

$$(ii) \quad x > y \text{ if and only if } \Delta^+ f(x) < \Delta^+ f(y),$$

$$(iii) \quad x > y \text{ if and only if } \Delta^+ f(x) < \Delta^- f(y) \text{ and } x \neq y,$$

$$(iv) \quad x > y+1 \text{ if and only if } \Delta^- f(x) < \Delta^+ f(y).$$

PROOF. The “only if” conditions for (i) and (ii) follow by analogous arguments as in the proof of Lemma A6 replacing discrete concavity with discrete strict concavity.

(i) If: Suppose to the contrary that $\Delta^-f(x) < \Delta^-f(y)$ and $x \leq y$. If $x < y$, then $\Delta^-f(x) > \Delta^-f(y)$ by the “only if” condition. If $x = y$, then $\Delta^-f(x) = \Delta^-f(y)$. In either case, we obtain a contradiction.

(ii) If: Follows analogously.

(iii) If: Suppose to the contrary that $\Delta^+f(x) < \Delta^-f(y)$ and $x < y$. Then by Lemma A6 (iv), $\Delta^+f(x) \geq \Delta^-f(y)$, a contradiction.

Only if: If $x > y$, apply (2.) from Lemma A5 to either (i) or (ii).

(iv) If: Suppose to the contrary that $\Delta^-f(x) < \Delta^+f(y)$ and $x \leq y + 1$. If $x = y + 1$, then $\Delta^-f(x) = \Delta^+f(y)$ by definition. If $x < y + 1$, then $\Delta^-f(x) > \Delta^+f(y)$ by (iii). In both cases, we have a contradiction.

Only if: Analogously to the proof of (iv) of Lemma A6 using discrete strict concavity instead of discrete concavity. $\square$

Corollary A1 *If f is discrete concave, then x satisfying “first-order conditions” is sufficient for x being a global maximizer.*

Discrete strict concavity is *not* sufficient for the global maximizer to be unique. A simple counterexample is $f(1) = 0 = f(4)$, $f(2) = f(3) = 1$, and $f(x) = -x$ for all other $x \in \mathbb{N}$. This function is easily shown to satisfy discrete strict concavity. However, the set of maximizers is $\{2, 3\}$. Nonetheless, the argmax cannot consist of more than two adjacent elements, as Lemma 7 in the main text establishes. More generally, we state:

Lemma A8 *If f is discrete strictly concave, then there are at most two global maximizers. If $\{x, y\}$ are the global maximizers, then either $x = y$ or $x = y + 1$. Moreover, if $x > y$, then x and y are both global maximizers if and only if $\Delta^-f(x) = 0 = \Delta^+f(y)$.*

PROOF. Suppose there exist $x, y \in \mathbb{N}_0$ such that first-order conditions are satisfied:

$$\begin{aligned}\Delta^+f(x) &\leq 0 \leq \Delta^-f(x), \\ \Delta^+f(y) &\leq 0 \leq \Delta^-f(y).\end{aligned}$$

By the previous corollary, both x and y must be global maximizers since f is discrete concave. It follows that:

$$\begin{aligned}\Delta^+f(x) &\leq 0 \leq \Delta^-f(y), \\ \Delta^+f(y) &\leq 0 \leq \Delta^-f(x).\end{aligned}$$

W.l.o.g., assume the $x > y$. If $x > y + 1$, then by Lemma A7 (iv), $\Delta^-f(x) < \Delta^+f(y)$, a contradiction to the last inequality. If $x = y + 1$, then $\Delta^-f(x) = \Delta^-f(y + 1) = \Delta^+f(y)$ by the definition of discrete derivatives. It follows that $\Delta^-f(x) = 0 = \Delta^+f(y)$.

Conversely, assume $x > y$ and $\Delta^-f(x) = 0 = \Delta^+f(y)$. Since f is discrete strictly concave, by Lemma A5 (1.), $\Delta^+f(x) < 0$ and $\Delta^-f(y) > 0$. Thus, the first-order conditions must hold. $\square$

For any multivariate function $f : \mathbb{N}^m \rightarrow \mathbb{R}$, we can analogously define the *partial discrete derivatives* by

$$\Delta_i^+ f(x_1, \dots, x_m) = f(x_1, \dots, x_{i-1}, x_i + 1, x_{i+1}, \dots, x_m) - f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_m)$$

and

$$\Delta_i^- f(x_1, \dots, x_m) = f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_m) - f(x_1, \dots, x_{i-1}, x_i - 1, x_{i+1}, \dots, x_m).$$

Finally, we will require as an ingredient to our discrete analysis the floor and ceiling functions defined by for $x \in \mathbb{R}$,

$$\begin{aligned} \lfloor x \rfloor &:= \max\{n \in \mathbb{Z} : n \leq x\} \\ \lceil x \rceil &:= \min\{n \in \mathbb{Z} : x \leq n\} \end{aligned}$$

We collect without proof well-known properties of the floor and ceiling functions:

Lemma A9 *For any $x, y \in \mathbb{R}$ and $n \in \mathbb{Z}$,*

- (i) $x - 1 \leq \lfloor x \rfloor \leq x \leq \lceil x \rceil \leq x + 1$
- (ii) $\lceil x \rceil = -\lfloor -x \rfloor$
- (iii) $\lceil x \rceil + \lceil y \rceil - 1 \leq \lceil x + y \rceil \leq \lceil x \rceil + \lceil y \rceil$
- (iv) $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1$
- (v) $\lfloor x + n \rfloor = \lfloor x \rfloor + n$
- (vi) $\lceil x + n \rceil = \lceil x \rceil + n$

B Log-concave Beliefs

Lemma B1 *Let $p \in \Delta(\Theta)$ be a full-support distribution. If p is log-concave (Assumption 2), then:*

- (i) *Relative likelihoods are non-increasing in i : For any $i, j = 1, \dots, m$ and k such that $j > i$ and $j + k \leq m$,*

$$\frac{p^{i+k}}{p^i} \geq \frac{p^{j+k}}{p^j}.$$

- (ii) *The discrete Mills' ratio is non-increasing in i : For any $i, j = 1, \dots, n$ with $j > i$, $m_d^i \geq m_d^j$, where:*

$$m_d^i := \frac{\sum_{k>i}^m p^k}{p^i}.$$

PROOF. For $i = 1, \dots, m - 2$,

$$\begin{aligned} (p^{i+1})^2 &\geq p^i p^{i+2} \\ \frac{p^{i+1}}{p^i} &\geq \frac{p^{i+2}}{p^{i+1}} \\ \log p^{i+1} - \log p^i &\geq \log p^{i+2} - \log p^{i+1}. \end{aligned}$$

Inductively, we have for $j > i$, $j + 1 \leq m$,

$$\log p^{i+1} - \log p^i \geq \log p^{j+1} - \log p^{j+1}.$$

(i) For any $i, j = 1, \dots, m$ and k such that $j + k \leq m$,

$$\begin{aligned} \frac{p^{i+k}}{p^i} &= \frac{p^{i+1}}{p^i} \frac{p^{i+2}}{p^{i+1}} \cdots \frac{p^{i+k}}{p^{i+k-1}} \\ \log \left(\frac{p^{i+k}}{p^i} \right) &= \log \left(\frac{p^{i+1}}{p^i} \frac{p^{i+2}}{p^{i+1}} \cdots \frac{p^{i+k}}{p^{i+k-1}} \right) \\ &= (\log p^{i+1} - \log p^i) + (\log p^{i+2} - \log p^{i+1}) + \dots + (\log p^{i+k} - \log p^{i+k-1}) \\ &\geq (\log p^{j+1} - \log p^j) + (\log p^{j+2} - \log p^{j+1}) + \dots + (\log p^{j+k} - \log p^{j+k-1}) \\ &= \log \left(\frac{p^{j+1}}{p^j} \frac{p^{j+2}}{p^{j+1}} \cdots \frac{p^{j+k}}{p^{j+k-1}} \right) = \log \left(\frac{p^{j+k}}{p^j} \right) \\ &= \frac{p^{i+1}}{p^i} \frac{p^{i+2}}{p^{i+1}} \cdots \frac{p^{i+k}}{p^{i+k-1}} = \frac{p^{i+k}}{p^i}, \end{aligned}$$

where the inequality follows from log-concavity applied to each term of the sum.

(ii) Rewrite (i) as:

$$p^j p^{i+k} - p^i p^{j+k} \geq 0.$$

Then,

$$\begin{aligned} \sum_{k=1}^{m-j} \left(p^j p^{i+k} - p^i p^{j+k} \right) &\geq 0 \\ p^j \left(\sum_{k>i}^{m-j+i} p^k \right) - p^i \left(\sum_{k>j}^m p^k \right) &\geq 0 \\ p^j \left(\sum_{k>i}^m p^k \right) - p^i \left(\sum_{k>j}^m p^k \right) &\geq 0. \end{aligned}$$

This establishes the lemma. □

While the previous lemma should be well-known, we were unable to locate a complete treatment of the finite case in the literature.

Lemma B2 Under Assumption 2, $\varphi^{i+1} < \varphi^i$ for all $i = 1, \dots, m - 1$.

PROOF. We have:

$$\begin{aligned} \varphi^{i+1} &< \varphi^i \\ \lceil \kappa^{(i+1)} \rceil + \frac{\sum_{j>i+1}^n p^j}{p^{i+1}} &< \lceil \kappa^{(i)} \rceil + \frac{\sum_{j>i}^n p^j}{p^i} \end{aligned}$$

follows directly from $\lceil \kappa^{(i+1)} \rceil < \lceil \kappa^{(i)} \rceil$ and Assumption 2. $\square$

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