A Note on Stochastic Orders and Incentives

Alejandro Francetich[*,†]

UW Bothell School of Business

October 15, 2025

Abstract

In contract design, a payoff-maximizing principal trades off social surplus for lower information rents. Imagine that the principal is able to influence the distribution of agent types; a monopolist can invest in marketing campaigns to boost demand, and a procurer can invest in expanding their search hoping to attract more-efficient contractors. Changes in the type distribution that generate more social surplus, however, may not be profitable for the principal if they lead to even higher information rents. *When is a social improvement in the distribution of agent types also profitable for the principal when accounting for the impact of information rents?*

In the context of linear utilities/costs, while the increasing convex order (resp., SOSD) establishes monotonicity of the social surplus when the agent is a buyer (resp., contractor), it is insufficient for private payoffs. We show that FOSD guarantees monotonicity of the principal's expected payoff—*even if the superior distribution needs to be ironed.*

Keywords: Mechanism design; screening; information rents; stochastic dominance

JEL Classification Numbers: D82, D86

[*]Email address: aletich@uw.edu

[†]I would like to thank Alejandro Manelli for his guidance and encouragement, as well as David Kreps and Jeremy Bulow for invaluable, insightful conversations. All errors and omissions are my own.

1 Introduction

A monopolist can invest in a marketing campaign to boost demand. If the buyer's reservation price is private information, a demand boost can take the form of a shift in the distribution of reservation prices: Ads may lead to shifting probability mass to higher values, or expanding the support upwards. Alternatively, a buyer seeking to procure from a contractor with private costs can invest in expanding the call for bids in the hopes of attracting more-efficient contractors.

In such contract-design problems, a payoff-maximizing principal trades off social surplus for lower information rents. Changes in the distribution of agent types may increase the social surplus generated, but they could also raise the amount due the agent in information rents. *When is a social improvement in the type distribution also profitable for the principal when accounting for the impact of information rents?*

In this note, we address this question in the context of a monopoly setting where the agent has linear utilities, and then adapt the main result to the procurement setting with linear costs. While the increasing convex order (resp., second-order stochastic dominance) corresponds to social improvements in the monopoly (resp., procurement) context, a stronger stochastic order is needed for the private and social benefits to align. We show that the principal's payoff is monotonic under first-order stochastic dominance—*even if the dominant distribution leads to a binding monotonicity constraint*, and thus entails ironing (Myerson, 1981).

The rest of the paper is organized as follows. We start with a monopoly example in Section 2. Section 3 describes the monopoly setting (subsection 3.1), discusses the first-best benchmark (subsection 3.2), and shows the monotonicity result for expected profit (subsection 3.3). In section 4, we adapt the result to the procurement setting. Finally, section 5 concludes.

2 Monopoly Setting

A widget manufacturer (she) produces and sells $q \geq 0$ widgets to a consumer (he) in exchange for payment $t \geq 0$. The cost of manufacturing widgets if given by $c(q) = \frac{q^2}{2}$, and so the seller's profit is $\pi(t, q) = t - \frac{q^2}{2}$. The consumer's valuation for widgets v is drawn uniformly from $[0, 1]$, and his payoff from trading with the seller is $u(q, t, v) = v \cdot q - t$.

Assume first that the consumer's valuation is publicly known. In this case, the seller can extract all of the consumer's surplus (his outside option being 0) and produce efficiently: $q^*(v) = v$. Thus, expected maximum profit, Π^*, equals expected maximum social surplus, W^*:

$$\Pi^* = W^* = \int_0^1 \frac{v^2}{2}\mathrm{d}v = \frac{1}{6}.$$

Imagine now that the seller can spend in advertising to boost the widget demand. As a result, the distribution of valuations shifts from cdf $F(v) = v$ to $F(v) = v^2$. While the change in distribution does not affect the allocation rule $q^*(v)$, it yields higher maximum profit/surplus:

$$\Pi^* = W^* = \int_0^1 \frac{v^2}{2} \cdot 2v\mathrm{d}v = \frac{1}{4}.$$

As long as the cost of the ad campaign is under $\frac{1}{12} \approx 0.0833$, the resulting demand boost is profitable.

But the widget manufacturer cannot observe her customer's valuation; she has to incentivize him through information rents. A buyer with valuation v commands rent in the amount of $\int_0^v q(x)\mathrm{d}x$. Trading off surplus for information rent, the seller now produces widgets according to the allocation rule $q^*(v) = \max\{2v - 1, 0\}$. Under the uniform distribution, her expected profit is:

$$\Pi^* = \int_{\frac{1}{2}}^1 \frac{(2v-1)^2}{2}\mathrm{d}v = \frac{1}{12} \approx 0.0833.$$

Under the shifted distribution, the new structure of information rents leads to the new production rule $q^*(v) = \max\left\{\frac{3v}{2} - \frac{1}{2v}, 0\right\}$; the corresponding profit is:

$$\Pi^* = \int_{\frac{1}{\sqrt{3}}}^1 \frac{1}{2}\left(\frac{3v}{2} - \frac{1}{2v}\right)^2 \cdot 2v\mathrm{d}v = \frac{\ln(3)}{8} \approx 0.1373.$$

For the campaign to remain profitable, the cost must be under $\frac{12\ln(3)-8}{96} \approx 0.0539$.

Now, let the valuation v be drawn from a $\mathrm{Beta}(5,5)$ distribution, and the seller can invest in shifting said distribution to a $\mathrm{Beta}(4,4)$. With valuations being publicly known, expected maximum profit is:

$$\Pi^* = \int_0^1 \frac{v^2}{2} \cdot v^4(1-v)^4\mathrm{d}v = \frac{3}{22} \approx 0.1364$$

before the campaign, and:

$$\Pi^* = \int_0^1 \frac{v^2}{2} \cdot v^3(1-v)^3 \mathrm{d}v = \frac{5}{36} \approx 0.1389$$

afterwards. The campaign is profitable provided its cost is under $\frac{1}{396} \approx 0.0025$.

Under asymmetric information, the allocation rules under the two distributions are presented in Figure 1. The corresponding maximum expected profits are $\Pi^* \approx 0.07139$ under the Beta$(5,5)$ distribution and $\Pi^* \approx 0.07108$ for $v \sim$ Beta$(4,4)$. Even if the campaign were free of charge, this boost in the demand for widgets leaves the seller *worse off*.

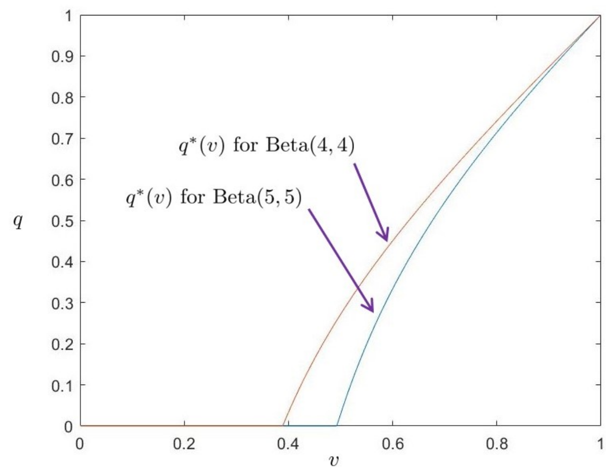

Figure 1: Allocation rules $q^*(v)$ for $v \sim$ Beta$(5,5)$ (blue) and $v \sim$ Beta$(4,4)$ (red).

In both scenarios, the shift in the distribution constitutes a *social improvement*—provided that the cost of switching is small enough. As such, they are also profitable for the seller under symmetric information. However, asymmetric information forces the seller to grant some rents to the consumer, which lead her to distort the allocation rule. The shift in the distribution of valuations increases the buyer's information rents and allocation distortion. In the second scenario, this effect dominates, leading to lower profit for the seller. Thus, the social improvement is *privately costly*.

As we show below, the difference in the two scenarios driving the result is that the first "distribution improvement" is an improvement in the sense of first-order stochastic dominance, while the second one is in the weaker sense of the increasing convex order.

3 Monopoly Setting

3.1 Environment

We expand on the monopoly setting from the example in Section 2. A monopolist (the principal, she) produces and sells quantity $q \geq 0$ of her product to a consumer (the agent, he) with valuation given by $v \geq 0$ in exchange for payment $t \geq 0$. The cost for the principal to produce q units is given by $c(q)$, a strictly increasing, concave, and twice-differentiable function.

There is a parametric family of probability distributions for the agent's valuation, $\mathcal{F} := \{F_\theta(v) : \theta \in \Theta\}$ for some parameter space Θ, with $F_\theta(v)$ denoting the cdf of the distribution identified by $\theta \in \Theta$. We assume that, for each $\theta \in \Theta$, F_θ is absolutely continuous and has full support on some interval $V_\theta := [\underline{v}_\theta, \overline{v}_\theta]$, where $0 \leq \underline{v}_\theta < \overline{v}_\theta$, with pdf $f_\theta(v)$.

Given $\theta \in \Theta$, the principal's problem is to design an implementable mechanism consisting of an allocation function $q : V_\theta \to [0, B]$, where $B > 0$ is a (fixed) capacity constraint, and a transfer function $t : V_\theta \to \mathbb{R}_+$ to maximize their expected profit:[1]

$$\Pi(q, t, \theta) := \int_{V_\theta} \pi(t(v), q(v)) f_\theta(v) \mathrm{d}v.$$

If we let $\mathcal{M}$ denote the space of implementable mechanisms, we are interested in the behavior of the function $\Pi^* : \Theta \to \mathbb{R}$ given by:

$$\Pi^*(\theta) := \sup(\{\Pi(q, t, \theta) : (q, t) \in \mathcal{M}\}).$$

Let $\succsim$ denote an order on $\mathcal{F}$; slightly abusing notation, we identify $\succsim$ with its projection on Θ and write $\theta' \succsim \theta$ for $F_{\theta'} \succsim F_\theta$. The goal is to find an order $\succsim$ on Θ such that $\Pi^*(\theta)$ is non-decreasing.

3.2 First Best and the Increasing Convex Order

If the agent's type, his valuation v, is publicly observed, the principal can extract all of the agent's surplus through the payment by setting $t(v) = v \cdot q(v)$. Thus, the monopolist's profit equals the social surplus, and the optimal allocation rule is the first-best allocation rule, $q^{FB}(v) := (c')^{-1}(v)$.

[1]We omit indexing the mechanism (q, t) by θ; nonetheless, the optimal mechanism will depend on θ, and the latter dependence will be made explicit.

Define the social surplus function $w : [0, B] \times \mathbb{R}_+ \to \mathbb{R}$ as $w(q, v) := v \cdot q - c(q)$, and the maximum social surplus function $w^* : \mathbb{R}_+ \to \mathbb{R}$ as $w^*(v) := w(q^{FB}(v), v)$. Then,

$$\Pi(q, t, \theta) = \int_{V_\theta} w(q(v), v) f_\theta(v) \mathrm{d}v \leq \int_{V_\theta} w^*(v) f_\theta(v) \mathrm{d}v =: W^*(\theta),$$

where $W^* : \Theta \to \mathbb{R}$ is the expected maximum social surplus function. It follows that $\Pi^*(\theta) = W^*(\theta)$. Notice that $w^{*'}(v) = (c')^{-1}(v)$, which makes $w^*(v)$ non-decreasing and convex. Thus, monotonicity of $W^*(\theta)$ is ensured under the increasing convex order.

Definition (Shaked and Shanthikumar, 2007)**.** For any $\theta, \theta' \in \Theta$, θ' dominates θ in the *increasing convex order* (ICx), $\theta' \succsim_{ICx} \theta$, if for any non-decreasing and convex function $\phi : \mathbb{R} \to \mathbb{R}$, the expectation of $\phi(v)$ under θ' is at least as high as under θ:

$$\int_{V_{\theta'}} \phi(v) f_{\theta'}(v) \mathrm{d}v \geq \int_{V_{\theta'}} \phi(v) f_{\theta'}(v) \mathrm{d}v.$$

Intuitively, if $\theta' \succsim_{ICx} \theta$, then the distribution identified by θ' has both a higher mean and a higher variance than its θ counterpart. Thus, the former has more mass on more extreme types in a way that raises the surplus. This order is the convex counterpart of second-order stochastic dominance. Shaked and Shanthikumar (2007) shows that $\theta' \succsim \theta$ if and only if:

$$\int_v^1 (1 - F_\theta(x)) \mathrm{d}x \leq \int_v^1 (1 - F_{\theta'}(x)) \mathrm{d}x \tag{1}$$

for every $v \in \mathbb{R}$ (Theorem 4.A.2).

It is immediate to see that if θ' first-order stochastically dominates θ, which we denote by $\theta' \succsim_{FO} \theta$, then $\theta' \succsim_{ICx} \theta$. Thus, $W^*(\theta)$ is also non-decreasing under $\succsim_{FO}$, as illustrated in the examples below.

Example 1. The monopolist's cost function is $c(q) = \frac{q^2}{2}$. Take $\Theta = [1, 2]$ and let $\theta \in \Theta$ identify the uniform distribution on $V_\theta = [0, \theta]$: $F_\theta(v) = \frac{v}{\theta}$ and $f_\theta(v) = \frac{1}{\theta}$. Here, $\theta' \geq \theta$ if and only if $\theta' \succsim_{FO} \theta$, and so $\theta' \succsim_{ICx} \theta$. We have $q^{FB}(v) = v$, $w^*(v) = \frac{v^2}{2}$, and:

$$\Pi^*(\theta) = W^*(\theta) = \int_0^\theta \frac{v^2}{2} \frac{1}{\theta} \mathrm{d}v = \frac{\theta^2}{6}.$$

Thus, $W^*(\theta)$ is strictly increasing. $\triangle$

Example 2. For the same monopolist, take now $\Theta = (-1, 1]$ and, for each $\theta \in \Theta$, let $V_\theta = [0, 1]$ and $F_\theta(v) = v^{1+\theta}$. Again, $\theta' \geq \theta$ if and only if $\theta' \succsim_{FO} \theta$. Maximum expected profit is:

$$W^*(\theta) = \int_0^1 \frac{v^2}{2}(1+\theta)v^\theta \mathrm{d}v = \frac{1+\theta}{6+2\theta},$$

which is also strictly increasing.

Example 3. Consider now the parameter space $\Theta = (1, +\infty)$. For each $\theta \in \Theta$, let $V_\theta = [0, 1]$ and $F_\theta(v) = 1 - (1-v)^{\frac{1}{\theta-1}}$. Notice that, for $\theta = 2$, this is simply the uniform distribution. As in the previous examples, $\theta' \geq \theta$ if and only if $\theta' \succsim_{FO} \theta$. Now,

$$W^*(\theta) = \int_0^1 \frac{v^2}{2} \frac{(1-v)^{\frac{2-\theta}{\theta-1}}}{\theta - 1} \mathrm{d}v = \frac{(\theta-1)^2}{\theta(2\theta-1)}.$$

See Figure 2. $\triangle$

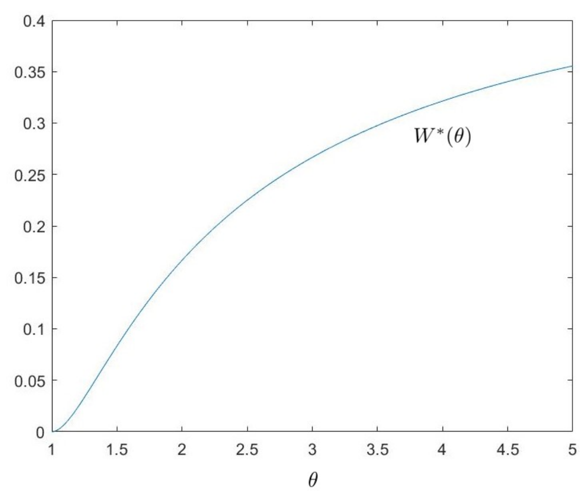

Figure 2: Curve for $W^*(\theta)$ in example 3.

3.3 Private information and FOSD

When the agent's type is his private information, the principal must award him information rents. For every $\theta \in \Theta$, implementability consists of the usual incentive compatibility and participation constraints, which are equivalent to the following conditions:[2]

- For every $v \in V_\theta$, $U(v) = \int_{\underline{v}_\theta}^{v} q(x)\mathrm{d}x$;
- The function $q : V_\theta \to [0, B]$ is non-decreasing.

With this characterization, we can write the principal's expected profit in terms of the allocation rule alone:

$$\Pi(q,\theta) = \int_{\underline{v}_\theta}^{\overline{v}_\theta} \left[u_\theta(v) \cdot q(v) - c(q(v))\right] f_\theta(v)\mathrm{d}v,$$

where $u_\theta(v) = v - \frac{1-F_\theta(v)}{f_\theta(v)}$ is the agent's virtual utility.

This last expression highlights the efficiency-rent trade-off that the principal faces: Generating more surplus can be profitable, but it can also mean more information rents for the agent. We can highlight this point by rewriting the expected profit formula as follows:

$$\Pi(q,\theta) = \int_{\underline{v}_\theta}^{\overline{v}_\theta} w(q(v), v) f_\theta(v)\mathrm{d}v - \int_{\underline{v}_\theta}^{\overline{v}_\theta} (1 - F_\theta(v))q(v)\mathrm{d}v.$$

Given an allocation rule $q(v)$ and provided that the function $w_q(v) := w(q(v), v)$ is non-decreasing, $\theta' \succsim_{FO} \theta$ implies both a higher expected surplus but also higher information rents: $1 - F_{\theta'}(v) \geq 1 - F_\theta(v)$.

For any given $\theta \in \Theta$, if $F_\theta(v)$ is *regular* (i.e., if $u_\theta(v)$ is non-decreasing), then the monotonicity constraint in the principal's maximization problem can be safely omitted. In this case, we find the optimal allocation rule by maximizing pointwise, leading to:

$$q^*_\theta(v) = (c')^{-1}\left(\max\{u_\theta(v), c'(0)\}\right). \tag{2}$$

If the regularity assumption fails, the optimal allocation rule is found by replacing $u_\theta(v)$ in (2) with $\bar{u}_\theta(v)$, the ironed virtual utility (Myerson, 1981).

Thus, letting $\tilde{u}_\theta$ equal u_θ if θ is regular and $\bar{u}_\theta$ otherwise, the principal's maximum expected profit is:

$$\Pi^*(\theta) := \int_{\underline{v}_\theta}^{\overline{v}_\theta} \left[\tilde{u}_\theta(v) \cdot q^*_\theta(v) - c(q^*_\theta(v))\right] f_\theta(v)\mathrm{d}v.$$

[2]See, for instance, Dewatripont and Bolton (2005); Krishna (2010); Laffont and Martimort (2001). Notice that we are already imposing the condition $U(\underline{v}_\theta) = 0$.

Recall the function $w^*(x) = (c')^{-1}(x) \cdot x - c\left((c')^{-1}(x)\right)$ and define the function $z(x) := \max\{x, c'(0)\}$. Both of these functions are non-decreasing and convex. They allow us to rewrite maximum expected profit as:

$$\Pi^*(\theta) = \int_{\underline{\theta}}^{\overline{v}_\theta} (w^* \circ z)\left(\tilde{u}_\theta(v)\right) f_\theta(v) \mathrm{d}v.$$

Notice that $w^* \circ z$ is also non-decreasing and convex.

Example 1 (Continued)**.** For the family of uniform distributions $U[0, \theta]$, we have $u_\theta(v) = 2v - \theta$. Maximum expected profit is:

$$\Pi^*(\theta) = \int_{\frac{\theta}{2}}^{\theta} \frac{(2v-\theta)^2}{2} \frac{1}{\theta} \mathrm{d}v = \frac{\theta^2}{12}.$$

Thus, $\Pi^*(\theta)$ is strictly increasing and strictly convex. $\triangle$

Example 2 (Continued)**.** For this family of distributions, the virtual utility is given by $u_\theta(v) = \frac{(2+\theta)v^{1+\theta}-1}{(1+\theta)v^\theta}$, which is strictly increasing. Defining $\lambda_\theta := (2+\theta)^{-\frac{1}{1+\theta}}$, we have:

$$\Pi^*(\theta) = \int_{\lambda_\theta}^{1} \frac{1}{2} \left[\frac{(2+\theta)v^{1+\theta}-1}{(1+\theta)v^\theta}\right]^2 (1+\theta)v^\theta \mathrm{d}v = \frac{1+\theta}{2(1-\theta)(\theta+3)} \left[1 - \left(\frac{1}{2+\theta}\right)^{\frac{1-\theta}{1+\theta}}\right].$$

Figure 3 depicts the curve of $\Pi^*(\theta)$. $\triangle$

Example 3 (Continued)**.** This family of distributions has affine virtual utilities: $u_\theta(v) = \theta v - \theta + 1$. Maximum expected profit for the monopolist is given by:

$$\Pi^*(\theta) = (\theta - 1)\theta^{-\frac{\theta}{\theta-1}}.$$

Figure 4 shows the $\Pi^*(\theta)$ curve. $\triangle$

For a given parametric family of distributions, even if virtual utilities are convex, the argument that establishes that $W^*(\theta)$ is non-decreasing with respect to $\succsim_{ICx}$ does not apply: The function $w^* \circ z \circ \tilde{u}_\theta$ changes with θ, reflecting the distortion in the allocation resulting from the distribution-specific structure of information rents. In fact, as seen in Section 2, the result is no longer true.

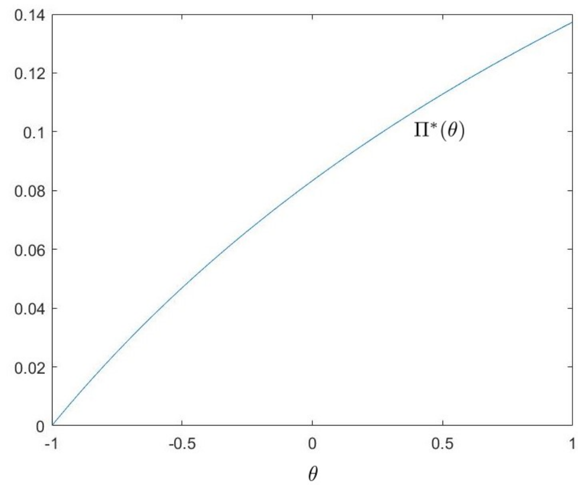

Figure 3: Curve for $\Pi^*(\theta)$ in Example 2.

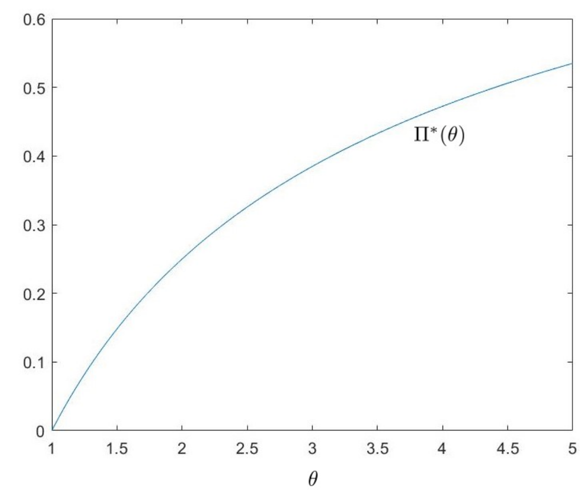

Figure 4: Curve for $\Pi^*(\theta)$ in Example 3.

Example 4 (Section 2)**.** Let $\Theta = \{1, 2\}$, with $\theta = 1$ being a Beta$(5, 5)$ distribution and $\theta = 2$ being a Beta$(4, 4)$ distribution. By (1), it can be checked that $2 \succsim_{ICx} 1$. However, our monopolist with cost function $c(q) = \frac{q^2}{2}$ finds $\theta = 1$ to be more profitable than $\theta = 2$: $\Pi^*(1) \approx 0.07139 > 0.07108 \approx \Pi^*(2)$. $\triangle$

In the earlier three examples, all maximum expected profits are non-decreasing in θ. As anticipated in Section 2, the key is that the families of distributions in said examples are ordered under $\succsim_{FO}$: $\theta' \geq \theta$ if and only if $\theta' \succsim_{FO} \theta$. It turns out that

$\succsim_{FO}$ is sufficient: Theorem 1 below shows that $\Pi^*(\theta)$ is non-decreasing with respect to $\theta' \succsim_{FO} \theta$.

To prove the theorem, it is convenient to change variables following Bulow and Roberts (1989): $x := 1 - F_\theta(v)$, $v_\theta(x) := F_\theta^{-1}(1-x)$. Notice that x has a uniform distribution on $[0,1]$ and that $v_\theta(x)$ is non-increasing. Next, define $R_\theta(x) := x \cdot v_\theta(x)$. This expression is analogous to a monopoly revenue formula, with x playing the role of quantity and $v_\theta(x)$ playing the role of inverse demand. We have that $R'_\theta(x) = u_\theta(v_\theta(x))$ and:

$$\Pi^*(\theta) = \int_0^1 (w^* \circ z) \left(R'_\theta(x)\right) \mathrm{d}x$$

if θ is regular. Otherwise, Myerson's process of ironing entails replacing $R_\theta(x)$ with its concave envelope, $\bar{R}_\theta(x)$:

$$\bar{R}_\theta(x) := \inf\left(\{r(x) : r : [0,1] \to \mathbb{R} \text{ is concave}, \forall \tilde{x} \in [0,1] \ (r(\tilde{x}) \geq R_\theta(\tilde{x}))\}\right).$$

By construction, $\bar{R}_\theta(x)$ is concave, so it is differentiable almost everywhere. Here,

$$\Pi^*(\theta) = \int_0^1 (w^* \circ z) \left(\bar{R}'_\theta(x)\right) \mathrm{d}x.$$

Theorem 1 (Monotonicity, monopoly setting)**.** *The maximum expected profit function $\Pi^*(\theta)$ is non-decreasing under $\succsim_{FO}$.*

Proof. We start with the case in which both distributions are regular. Pick any $\theta, \theta' \in \Theta$ such that $\theta' \succsim_{FO} \theta$; we have $v_{\theta'}(x) \geq v_\theta(x)$ for all $x \in [0,1]$:

$$F_\theta(v_\theta(x)) = 1 - x = F_{\theta'}(v_{\theta'}(x)) \leq F_\theta(v_{\theta'}(x)).$$

Thus, since $R_{\theta'}(0) = R_\theta(0) = 0$,

$$\int_0^x R'_{\theta'}(y)\mathrm{d}y = R_{\theta'}(x) \geq R_\theta(x) = \int_0^x R'_\theta(y)\mathrm{d}y.$$

Regularity means that $R_{\theta'}(x)$ and $R_\theta(x)$ are concave. Dominance means that $R_{\theta'}(x) \geq R_\theta(x)$ for all $x \in [0,1]$; or, alternatively, that $R_{\theta'}$ weakly submajorizes R_θ (Hardy, Littlewood, and Pólya, 1952; Marshall, Olkin, and Arnold, 2011).[3] An extension of the Hardy-Littlewood-Pólya inequality from Chong (1974) (Theorem 2.3) states that

[3]For non-decreasing functions, their decreasing rearrangement yields the original function.

$R_{\theta'}(x) \geq R_\theta(x)$ is thus equivalent to the condition that:

$$\int_0^1 h\left(R'_{\theta'}(x)\right) \mathrm{d}x \geq \int_0^1 h\left(R'_\theta(x)\right) \mathrm{d}x$$

for all non-decreasing and convex functions $h : \mathbb{R} \to \mathbb{R}$. Since $(w^* \circ z)$ satisfies said properties, the result follows:

$$\Pi^*(\theta') = \int_0^1 (w^* \circ z)\left(R'_{\theta'}(x)\right) \mathrm{d}x \geq \int_0^1 (w^* \circ z)\left(R'_\theta(x)\right) \mathrm{d}x = \Pi^*(\theta).$$

If one or both of the distributions is not regular, the corresponding virtual utility must be ironed. We start with the case where θ is regular but θ' is irregular. We have that $\bar{R}_{\theta'}(x) \geq R_{\theta'}(x) \geq R_\theta(x)$ for all $x \in [0,1]$. Since $R_{\theta'}(x)$ is concave, it is differentiable almost everywhere and—along with the fact that $\bar{R}_\theta(0) = 0$—we have that:

$$\bar{R}_{\theta'}(x) = \int_0^x \bar{R}'_{\theta'}(y)\mathrm{d}y.$$

Thus, $\bar{R}_{\theta'}(x)$ weakly submajorizes R_θ, and the rest of the argument follows as above. If θ is irregular but θ' is regular, weak submajorization follows from the observation that $R_{\theta'}(x)$ is in the set that defines $\bar{R}_\theta(x)$, so $R_{\theta'}(x) \geq \bar{R}_\theta(x)$ for all $x \in [0,1]$. Finally, combining the two cases tackles the case when both θ and θ' are irregular.

Alternatively, following Roughgarden and Schrijvers (2015), ironing the virtual utility is equivalent to replacing the irregular distribution $F_{\theta'}$ with a regular one $\bar{F}_{\theta'}$ such that: (1) $\bar{F}_{\theta'} \succsim_{FO} F_{\theta'}$; (2) The ironed virtual utility for $F_{\theta'}$ is identical to the virtual utility for $\bar{F}_{\theta'}$; and (3) The expected profit given the ironed allocation rule under $F_{\theta'}$ and $\bar{F}_{\theta'}$ are equal. The Lemma in the appendix shows that, for any regular type distribution G, $F_{\theta'} \succsim_{FO} G$ implies $\bar{F}_{\theta'} \succsim_{FO} G$ and $G \succsim_{FO} F_{\theta'}$ implies $G \succsim_{FO} \bar{F}_{\theta'}$. Thus, the previous argument applies, and it can be easily adapted to when F_θ is the irregular distribution, or when both are irregular. $\square$

4 Procurement Setting

This section reverses the roles: The principal is now the buyer and the agent is the producer whose type, c, represents his marginal/average cost. The principal enjoys a utility $u(q,t) = v(q) - t$, where $v(q)$ is strictly increasing, strictly concave, and twice continuously differentiable. Now, expected social surplus and principal's payoff are:

$$\begin{aligned} W(q,\theta) &:= \int_{\underline{c}_\theta}^{\bar{c}_\theta} [\nu(q(c)) - c\cdot q(c)]\, f_\theta(c)\mathrm{d}c, \\ P(q,\theta) &:= \int_{\underline{c}_\theta}^{\bar{c}_\theta} [\nu(q(c)) - k_\theta(c)\cdot q(v)]\, f_\theta(c)\mathrm{d}c, \end{aligned}$$

where $k_\theta(c) = c + \frac{F_\theta(c)}{f_\theta(c)}$ is the agent's virtual cost.

The first-best allocation rule is $q^{FB}(c) = (\nu')^{-1}(c)$, which is non-increasing. We now say that a type distribution is regular if $k_\theta(c)$ is non-decreasing. Writing $\tilde{k}_\theta(c)$ for $k_\theta(c)$ if θ is regular and for the ironed virtual cost $\bar{k}_\theta(c)$ otherwise,[4]

$$q^*_\theta(c) = (\nu')^{-1}\left(\max\left\{\tilde{k}_\theta(c), \nu'(0)\right\}\right).$$

The maximum social surplus given c is $w^*(c) := \nu\left(q^{FB}(c)\right) - c\cdot q^{FB}(c)$, which is non-increasing and convex. To establish monotonicity of $W^*(\theta) := \int_{\underline{c}_\theta}^{\bar{c}_\theta} w^*(c) f_\theta(c)\mathrm{d}c$, we invoke second-order stochastic dominance, $\succsim_{SO}$.[5] Since the function $-w^*(c)$ is non-decreasing and concave, $\theta' \succsim \theta$ implies that:

$$\int_{\underline{c}_\theta}^{\bar{c}_\theta} [-w^*(c)] f_{\theta'}(c)\mathrm{d}c \geq \int_{\underline{c}_\theta}^{\bar{c}_\theta} [-w^*(c)] f_\theta(c)\mathrm{d}c,$$

or $W^*(\theta') \geq W^*(\theta)$. Thus, $W^*(\theta)$ is non-increasing under $\succsim_{SO}$.

Denote the principal's maximum expected payoff as $P^*(\theta)$; we have:

$$P^*(\theta) = \int_0^1 (w^* \circ z)\left(\tilde{k}_\theta(c_\theta(x))\right)\mathrm{d}x.$$

Example 5. Let $\Theta = [0,1]$ and let $\theta \in \Theta$ represent the uniform distribution on $[\theta, 2]$: $F_\theta(c) = \frac{c-\theta}{2-\theta}$. We have $k_\theta(c) = 2c - \theta$, so all the distributions in the family are regular, and $\theta' > \theta$ means $\theta' \succsim_{FO} \theta$. The principal's utility is $\nu(q) = 100q - q^2$ (assuming that $q \leq 100$), so $q^*_\theta(c) = 100 - 2c + \theta$. The principal's maximum expected payoff is:

$$P^*(\theta) = \int_\theta^2 \frac{1}{2}\cdot(100 - 2c + \theta)^2 \cdot \frac{1}{2-\theta}\mathrm{d}c = \frac{\theta^2 - 4\theta + 28,816}{6},$$

which is decreasing.

[4]The procurement setting is a separable case in the sense of Toikka (2011).

[5]In Shaked and Shanthikumar (2007), this order is called the increasing concave order.

The counterpart of Theorem 1 is below.

Theorem 2 (Monotonicity, procurement setting)**.** *The maximum expected payoff function $P^*(\theta)$ is non-increasing under $\succsim_{FO}$.*

Proof. Assume first that θ, θ' are regular distributions. We can write the principal's maximum expected payoff as:

$$P^*(\theta) = \int_0^1 h\left(-\tilde{k}_\theta(c_\theta(x))\right) \mathrm{d}x,$$

where $h(y) := (w^* \circ z)(-y)$ for $y < 0$. Since $w^* \circ z$ is non-increasing and convex, h is non-decreasing and convex.[6] For $\theta' \succsim_{FO} \theta$, we have $R_{\theta'}(x) := x \cdot F_{\theta'}^{-1}(x) \geq x \cdot F_\theta^{-1}(x) =: R_\theta(x)$. We have that $R'_\theta(x) = k_\theta(c_\theta(x)) \geq 0$ and similarly for θ', so:

$$\int_0^x R'_{\theta'}(y)\mathrm{d}y \geq \int_0^x R'_\theta(y)\mathrm{d}y$$
$$\int_0^x [-R'_{\theta'}(y)]\mathrm{d}y \leq \int_0^x [-R'_\theta(y)]\mathrm{d}y.$$

By the same argument as in the main theorem, it follows that:

$$P^*(\theta') = \int_0^1 h\left(-R'_{\theta'}(x)\right) \mathrm{d}x \leq \int_0^1 h\left(-R'_\theta(x)\right) \mathrm{d}x = P^*(\theta).$$

If ironing is needed, we simply replace $R_\theta(x)$, $R_{\theta'}(x)$ with their convex envelope, as in the proof of Theorem 1. Thus, the result follows. □

5 Conclusion

In this note, we raise the question of how a change in the distribution of the agent's type impacts the principal's expected payoff. In particular, we consider when a "social improvement" in the type distribution is also "privately" beneficial for a principal trading off social surplus for lower information rents.

As with externalities, information rents mean that socially-beneficial shifts in the distribution of agent types may lead to a drop in the principal's expected payoff. The increasing convex order (resp., second-order stochastic dominance) corresponds to social improvements in the monopoly (resp., procurement) context, but a stronger

[6]Although w^* in this section is not isotone, the composition $w^* \circ z$ is constant on $(-\infty, 0)$ and equals w^* on $[0, +\infty)$.

order is needed for the private and social incentives to align. First-order stochastic dominance does the trick—even when the dominant distribution entails ironing.

Avenues for future research are extending the results for the cases where the agent has quasilinear utilities/non-linear costs, and in dynamic mechanisms.

A Appendix

The lemma below establishes some properties of the "ironed" distribution from Roughgarden and Schrijvers (2015). For the sake of completeness, we include the construction of said distribution and some of its basic properties expressed in our setting.

Lemma (Ironed distributions)**.** *Let F be the cdf of an absolutely continuous and full-support distribution on an interval $[\underline{v}, \overline{v}]$. There exists an absolutely continuous, full-support, and regular distribution $\bar{F}$ on $[\underline{v}, \overline{v}]$ such that:*

1. *$\bar{F} \succsim_{FO} F$ and $\bar{F} = F$ outside of the bunching interval;*
2. *The ironed virtual utility for F, $\overline{u}_F(v)$, equals the virtual utility for $\bar{F}$, $u_{\overline{F}}(v)$;*
3. *$\int_0^1 (h \circ z)\left(\bar{R}'(x)\right) \mathrm{d}x = \int_0^1 (h \circ z)\left(R'_{\bar{F}}(x)\right) \mathrm{d}x$;*
4. *For any regular type distribution G, $F \succsim_{FO} G$ implies $\bar{F} \succsim_{FO} G$;*
5. *For any regular type distribution G, $G \succsim_{FO} F$ implies $G \succsim_{FO} \bar{F}$.*

Proof. Take $R(x)$ and its concave upper envelope, $\bar{R}(x)$. Define $\overline{v}(x) : [0,1] \to [\underline{v}, \overline{v}]$ as $\overline{v}(x) = \frac{\bar{R}(x)}{x}$ for $x > 0$ and $\overline{v}(0) := \lim_{x \to 0^+} \frac{\bar{R}(x)}{x}$. Since $\bar{R}(x) \geq R(x)$, we have that $\overline{v}(x) \geq v(x)$, with equality outside the bunching interval $[x_1, x_2]$, where $x_1 = 1 - F(v_1)$ and $x_2 = 1 - F(v_2)$. Moreover, $\overline{v}(x)$ is of the form $m + \frac{b}{x}$—since $\bar{R}(x)$ is affine—on the bunching interval, where $b > 0$ and where $m \geq 0$ is the ironed value of the ironed virtual utility. Define $\bar{F}$ as the cdf whose quantiles correspond to $\overline{v}(x)$:

$$\bar{F}(v) = 1 - \inf\left(\{x \in [0,1] : v \geq \overline{v}(x)\}\right).$$

To show 1, notice that the inequality $\overline{v}(x) \geq v(x)$ implies $\{x \in [0,1] : v \geq \overline{v}(x)\} \subseteq \{x \in [0,1] : v \geq v(x)\}$, and so:

$$\inf\left(\{x \in [0,1] : v \geq \overline{v}(x)\}\right) \geq \inf\left(\{x \in [0,1] : v \geq \overline{v}(x)\}\right) = 1 - F(v).$$

So, $\bar{F}(v) \leq 1 - (1 - F(v)) = F(v)$. Next, take any v outside of the bunching interval; for the corresponding $x = 1 - F(v)$, we have that $\overline{v}(x) = v(x)$, so the result follows.

To establish 2, we need only to look at the bunching interval, where $\overline{v}(x) = m + \frac{b}{x}$. Thus, the inequality $v \geq \overline{v}(x)$ can be rewritten as $x \geq \frac{b}{v-m}$, since $v > m$ on the bunching interval (due to downward distortion). Thus, $\bar{F}(v) = 1 - \frac{b}{v-m}$.[7] Straightforward computation shows that $u_{\overline{F}}(v) = m = \overline{u}_F(v)$.

Part 3 is a special case of a result in Myerson (1981), although in the present context it is an immediate consequence of the fact that $\bar{R}'(x) = R'(x)$ for every $x \in [0,1]$.

For 4, let G be the cdf of a regular distribution that $G \succsim_{FO} F$. Regularity implies that $R_G(x)$ is concave, while dominance implies that $R_G(x) \geq R(x)$ for every $x \in [0,1]$. Thus, $R_G(x) \geq \bar{R}(x)$, which implies that $G \succsim_{FO} \bar{F}$.

For 5, let G be the cdf of a regular distribution such that $F \succsim_{FO} G$. Then, $R(x) \geq R_G(x)$ for every $x \in [0,1]$, so $\bar{R}(x) \geq R_G(x)$ follows immediately from 1. $\square$

References

Bulow, J. and Roberts, J. (1989). The Simple Economics of Optimal Auctions. *The Journal of Political Economy*, Vol. 97, No. 5., pp. 1060-1090.

Chong, K-M. (1974). Some Extensions of a Theorem of Hardy, Littlewood and Pólya and Their Applications. *Canadian Journal of Mathematics*, 26(6): 1321-1340.

Dewatripont, M. and Bolton, P. (2005). Contract Theory. *MIT Press*.

Hardy, G., Littlewood, J.E., and Pólya, G. (1952). Inequalities. *Cambridge Mathematical Library*. 2nd edition.

Krishna, V. (2010). Auction Theory. *Academic Press*. 2nd edition.

Laffont, J.J. and Martimort, D. (2001). The Theory of Incentives: The Principal-Agent Model. *Princeton University Press*.

Marshall, A.W., Olkin, I., and Arnold, B.C. (2011). Inequalities: Theory of Majorization and Its Applications. *Springer Series in Statistics*. 2nd edition.

Milgrom, P. (2004). Putting Auction Theory to Work. *Cambridge University Press*.

[7] Alternatively, this expression for $\bar{F}$ on the bunching interval can be obtained as the solution to the ODE $\frac{1-F(v)}{f(v)} = v - m$ with boundary conditions $\bar{F}(v_1) = F(v_1)$ and $\bar{F}(v_2) = F(v_2)$.

Myerson, R. (1981). Optimal auction design. *Mathematics of Operations Research* 6(1): 58–73.

Roughgarden, T. and Schrijvers, O. (2015) Ironing in the Dark. `https://arxiv.org/abs/1511.06918`

Shaked, M. and Shanthikumar, J. G. (2007). Stochastic Orders. *Springer Series in Statistics*.

Toikka, J. (2011). Ironing without control. *Journal of Economic Theory* 146: 2510–2526.